CONVEXITY AND CURVATURE IN LORENTZIAN GEOMETRY

BY

WILLIAM A. KARR

DISSERTATION

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Doctoral Committee:

Professor Jeremy Tyson, Chair
Professor Emeritus Stephanie B. Alexander, Director of Research
Professor Emeritus Richard L. Bishop
Professor Christopher Leininger

Abstract

A space-time satisfies $\mathcal{R} \geq K$ if the sectional curvatures are bounded below by K for spacelike planes and above by K for timelike planes (similarly, a space-time satisfies $\mathcal{R} \leq K$ if the aforementioned inequalities are reversed). We demonstrate that these curvature bound conditions together with convex functions are effective means to study the geometry of space-times.

Chapter 3 explores the relation between convex functions and geodesic connectedness of space-times. We give geometric-topological proofs of geodesic connectedness for classes of space-times to which known methods do not apply. For instance, a null-disprisoning space-time is geodesically connected if it supports a proper, nonnegative strictly convex function whose critical set is a point. In particular, timelike strictly convex hypersurfaces of Minkowski space (which are prototypical examples of space-times satisfying $\mathcal{R} \geq 0$) are geodesically connected.

Chapter 4 explores the relationship between so-called λ-convex functions ($\nabla^2 f(x,x) \geq \lambda\langle x,x\rangle$), curvature bounds, and trapped submanifolds. We show that certain types of trapped submanifolds can be ruled out for domains of space-times satisfying $\mathcal{R} \leq K$. Using the full curvature bound condition $\mathcal{R} \leq K$ allows us to extend previous results that use timelike sectional curvature bounds to rule out trapped submanifolds in the chronological future of a point.

To my grandparents William & Carol Karr
and my great grandparents Douglas & Sylvia Morley.

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Chapter 1

Introduction

1.1 Curvature bounds in semi-Riemannian manifolds

In 1979, R. Kulkarni proved that at any point in a semi-Riemannian manifold with an indefinite metric, if the sectional curvatures are bounded above or below, then the sectional curvature at that point is constant [K79]. As a result, for many years there was little interest in extending the theory of Riemannian manifolds with sectional curvature bounded above or below to the Lorentzian and semi-Riemannian cases using all sectional curvatures and all geodesics.

Most of the work that followed using curvature bounds to study the global geometry of space-times considered only timelike geodesics and bounds on timelike sectional curvatures or Ricci curvature on timelike vectors. One of the most important results in Lorentzian geometry is the splitting theorem due to Eschenburg and many other researchers, the Lorentzian analog of the Cheeger-Gromoll splitting theorem for Riemannian manifolds of nonnegative Ricci curvature [BEE96, p. 506]. The Lorentzian splitting theorem states that if (M, g) is a globally hyperbolic space-time with dimension > 2 satisfying the strong energy condition ($\mathrm{Ric}(v, v) \geq 0$ when $g(v, v) < 0$) and containing a timelike line, then (M, g) is isometric to $(\mathbb{R} \times V, -dt^2 + h)$ where (V, h) is a complete Riemannian manifold. As another example of global Lorentzian curvature comparison, S.G. Harris proved a global triangle comparison theorem for timelike geodesics in globally hyperbolic space-times whose timelike sectional curvatures are bounded above [BEE96, App. A].

The curvature comparisons we shall consider are of the following type. A semi-Riemannian manifold is said to satisfy $\mathcal{R} \geq K$ if the spacelike sectional curvatures are bounded below by K and the timelike ones are bounded *above* by K. Equivalently, the curvature tensor R satisfies

$$g(R(X, Y)X, Y) \geq K(g(X, X)g(Y, Y) - g(X, Y)^2)$$

for all pairs of vector fields X, Y on M. Similarly, we define semi-Riemannian manifolds satisfying $\mathcal{R} \leq K$ by reversing the inequalities. Spaces satisfying these curvature bounds are abundant and include the big bang cosmological models discussed by Hawking and Ellis [HE93, p. 134-138]. In fact, the curvature bound is one third of the cosmological constant of the model universe [AB08, Ex. 7.3, p. 278]! Andersson and Howard introduced the curvature conditions $\mathcal{R} \geq K$ and $\mathcal{R} \leq K$ in [AH98]. Alexander and Bishop found its geometric significance by proving that it is equivalent to local Alexandrov triangle comparisons on the *signed* lengths of geodesics [AB08, Theorem 1.1]. The results of Alexander and Bishop suggest that methods from Alexandrov geometry can be used to take a more geometric approach to studying geodesics and global structure of space-times and semi-Riemannian manifolds.

1.2 Geodesic connectedness of Lorentzian manifolds

An outstanding problem that has been studied in Lorentzian geometry is geodesic connectedness. A space-time is geodesically connected if any pair of points can be joined by a geodesic (where by *geodesic*, we mean a curve satisfying the geodesic equation $\nabla_{\gamma'}\gamma' = 0$). In Riemannian geometry, the Hopf-Rinow theorem ensures that all complete connected Riemannian manifolds are geodesically connected. There is no analog to the Hopf-Rinow theorem for Lorentzian or semi-Riemannian manifolds.

Geodesic connectedness is a highly non-trivial property of space-times since not even de Sitter space, the model space of constant curvature $K > 0$, is geodesically connected (see Example 2.2.2). A primary goal of Alexandrov geometry is to produce globalization theorems for arbitrary triangles in spaces satisfying Alexandrov curvature bounds. Such theorems only apply to domains of spaces with geodesics connecting each pair in any triple of points.

A Lorentzian manifold (M, g) is an *orthogonal splitting space-time* if for a Riemannian manifold (M_0, g_0), M is isometric to $M_0 \times \mathbb{R}$ with the metric $g_0(A(p,t)\cdot, \cdot) - \beta(p,t)dt^2$ where $A(p,t) : T_pM_0 \to T_pM_0$ is a smoothly varying, symmetric, strictly positive linear operator and $\beta : M_0 \times \mathbb{R} \to \mathbb{R}$ is a smooth positive function. Sufficient conditions for geodesic connectedness of Lorentzian manifolds are given by an early theorem of Uhlenbeck [Uh75, Theorem 5.3], and by [BEE96, Theorem 11.25]. Both of these theorems concern spaces with no conjugate points. Uhlenbeck uses what she calls the metric growth condition for a Lorentzian manifold with respect to some orthogonal splitting. She proves a Hadamard-Cartan theorem for Lorentzian manifolds which states that a globally hyperbolic Lorentzian manifold satisfying a metric growth condition and having a condition on "null sectional curvatures" can be covered by a space which is topologically Minkowski space. She remarks [Uh75, p. 75] that the metric growth condition is not very satisfactory since different orthogonal splittings of a Lorentzian manifold may have vastly different structure. She then makes a plea for a more geometric condition under which the same results could be found.

At the end of the 1980s, Benci, Fortunato, Giannoni, and Masiello studied geodesic connectedness via an infinite-dimensional variational theory (see [M94, M06] and references therein). For Lorentzian manifolds carrying a timelike or null Killing field, geodesic connectedness has only recently become well understood [CFS08, BCF14]. It is also known to hold for globally hyperbolic space-times satisfying time-dependent orthogonal splittings under certain conditions, as summarized in Theorem 3.8.2 in Chapter 3. An informative survey of geodesics in semi-Riemannian manifolds by Candela and Sanchez outlines most of the progress in this area and they make a similar remark [CS08, Remark 4.47] to Uhlenbeck's: "it should be more interesting to obtain" a geodesic connectedness result "under weaker hypotheses on the metric or under intrinsic hypotheses related to the geometry of the manifold."

In an early and influential consideration of geodesic connectedness for Riemannian manifolds, Gordon proved that if a connected Riemannian manifold M supports a proper, nonnegative convex function, then M is geodesically connected [Go74]. Chapter 3 below examines the relationship between the existence of classically convex functions on space-times and geodesic connectedness. We prove a semi-Riemannian version of Gordon's theorem.

We also prove that the timelike strictly convex hypersurfaces of Minkowski space are geodesically connected. Our result uses geometric hypotheses to establish geodesic connectedness and does not require the existence of orthogonal splittings with special growth conditions. We exhibit simple examples of timelike convex hypersurfaces for which no natural orthogonal splittings seem to exist satisfying the metric growth conditions cited by Candela and Sanchez.

Timelike convex hypersurfaces satisfy $\mathcal{R} \geq 0$. Thus our theorem on geodesic connectedness of convex hypersurfaces may indicate that curvature bounds could serve as a geometric means to study geodesic connectedness of Lorentzian and semi-Riemannian manifolds.

1.3 Convex functions, curvature bounds, and trapped submanifolds

In 1965, Penrose introduced the notion of a trapped surface, i.e. a two dimensional closed spacelike submanifold in four dimensional space-time for which the so-called *null expansions* are negative [Pe65]. The region inside the Schwarzchild radius surrounding a black hole is foliated with trapped surfaces. The notion of a trapped surface was integral in the proofs of the classical singularity theorems and in the study of cosmic censorship [HE93, HP70, Pe65, Sen98].

The notion of a trapped surface can be generalized to that of a trapped submanifold whose codimension is arbitrary. Instead of using null expansions, trapped submanifolds are characterized by their mean curvature vector fields (see Definition 4.6.3). Galloway and Senovilla more recently proved analogs of the standard singularity theorems for Lorentzian manifolds of arbitrary dimension if they contain closed trapped submanifolds of arbitrary codimension [GS10].

Gibbons and Ishibashi initiated a study of the possible uses of convex functions in General Relativity in [GI01]. Gibbons and Ishibashi introduce and mainly consider "space-time convex" functions on Lorentzian manifolds (M, g). A function $f : M \to \mathbb{R}$ is space-time convex if $\nabla^2 f \geq \lambda g, \lambda > 0$ and $\nabla^2 f$ has Lorentzian signature. They show that the domains of space-time convex functions cannot contain closed marginally inner and outer trapped surfaces. Curvature bounds do not arise in their considerations of how to construct space-time convex functions.

In [EGK03], Erkekoglu, García-Río, and Kupeli proved Hessian and Laplacian comparison theorems for the level sets of the Lorentzian distance function from a point and from an achronal spacelike hypersurface in two space-times M and $\widetilde{M}$. In [AHP10], Alías, Hurtado, and Palmer study the restriction of the Lorentzian distance function to a point or spacelike hypersurface satisfying the Omori-Yau maximum principle. Under constant bounds either above or below, they establish sharp estimates on the mean curvature of such hypersurfaces. In [Imp12], Impera studies Hessian and Laplacian comparisons for the Lorentzian distance to a point, assuming timelike sectional curvature bounds above or below. Finally, in [ABL16], Alías, Bessa, and de Lira prove non-existence results and sharp mean curvature estimates for trapped submanifolds (of arbitrary codimension), based on the comparison inequalities for the Laplacian of the restriction to a spacelike submanifold of the Lorentzian distance functions examined in the aforementioned works. In particular, they use timelike sectional curvature bounds below to rule out future trapped submanifolds in domains lying in the chronological future of a point.

To our knowledge, the work in [ABL16] is the first attempt to use (timelike) sectional curvature bounds to rule out the existence of trapped submanifolds in space-times. It turns out that many of the Hessian comparisons established in [EGK03, AHP10, Imp12] can be extended to shape operator comparisons for level sets of signed distance functions in semi-Riemannian manifolds satisfying $\mathcal{R} \leq K$ or $\mathcal{R} \geq K$ as established by Andersson and Howard in [AH98] and Alexander and Bishop in [AB08]. In Chapter 4, we use the curvature bound condition of $\mathcal{R} \leq K$ to rule out certain types of trapped submanifolds in domains that extend those used in [ABL16].

Chapter 2

Preliminaries

2.1 Terminology

In this section we review basic Lorentzian and semi-Riemannian geometry and causality theory. Our references are [BEE96, HE93, O'N83].

Definition 2.1.1. A *semi-Riemannian manifold* (M, g) is a connected manifold M with a smoothly varying non-degenerate not-necessarily-positive-definite symmetric bilinear form g. A *Lorentzian manifold* is a semi-Riemannian manifold whose metric g has signature $(+, ..., +, -)$.

As with Riemannian manifolds, the metric determines a unique torsion-free connection satisfying

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for any smooth vector fields X, Y, Z on M. This connection is the *Levi-Civita connection* of (M, g) and satisfies all the same formal relations as in Riemannian geometry even when the metric is not positive definite.

The definitions of the curvature tensor, Ricci curvature, scalar curvature, and sectional curvature all may be defined as in the Riemannian case, except the sectional curvature is undefined on planes on which the restriction of g is degenerate.

Definition 2.1.2. A *geodesic* in M is a curve $\gamma : I \to M$ satisfying the *geodesic equation*

$$\gamma'' = \nabla_{\gamma'}\gamma' = 0. \tag{2.1}$$

The exponential map is defined exactly as it is in Riemannian geometry, i.e. for v in a sufficiently small neighborhood of $0 \in T_pM$, $\exp_p(v) = \gamma(1)$, where γ is unique inextendible geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Here an inextendible curve is one whose image contains all of its limit points in M.

Definition 2.1.3. For a C^1 function $f : M \to \mathbb{R}$, we can define the *gradient vector field* as the unique vector field ∇f satisfying

$$g(X, \nabla f) = Xf$$

for any smooth vector field X on M.

Definition 2.1.4. For a C^2 function $f : M \to \mathbb{R}$, the *Hessian of f* is the symmetric $(0,2)$ tensor field $\nabla^2 f$ defined by

$$\nabla^2 f(X,Y) = X(Yf) - (\nabla_X Y)f \tag{2.2}$$

for vector fields X, Y on M. Equivalently, $\nabla^2 f(X,Y) = g(\nabla_X \nabla f, Y)$. For any geodesic γ, $(f \circ \gamma)''(t) = \nabla^2 f(\gamma'(t), \gamma'(t))$.

A C^2 function $f : M \to \mathbb{R}$ is *convex* if $\nabla^2 f(x,x) \geq 0$ for all tangent vectors $x \in T_pM$, $p \in M$. Equivalently, $(f \circ \gamma)''(t) \geq 0$ for all geodesics $\gamma : I \to M$. We say that f is *strictly convex* if the above inequalities are all strict except when $x = 0$ or γ is constant. In Chapter 3, we investigate the relationship between semi-Riemannian manifolds supporting convex functions and geodesic connectedness.

For a smooth positive function $\lambda : M \to \mathbb{R}$, f is *λ-convex* if for all $p \in M$ and $x \in T_pM$, $\nabla^2 f(x,x) \geq \lambda\, g(x,x)$. Note that if M is not Riemannian, a λ-convex function is not necessarily convex. Locally, the conditions $\mathcal{R} \geq K$ and $\mathcal{R} \leq K$ are equivalent to a differential inequality satisfied by a modified signed distance function [AB08, Corollary 4.6]. In the latter case, these functions are λ-convex and can be used to rule out certain types of trapped submanifolds in their domains (see Theorems 4.5.5 and 4.6.5).

Definition 2.1.5. Let $\Sigma \subset M$ be a k-dimensional semi-Riemannian submanifold of M. Denote the Levi-Civita connections on M and Σ by $\overline{\nabla}$ and ∇, respectively. We define the *second fundamental form* $\mathrm{II} : T_p\Sigma \times T_p\Sigma \to (T_p\Sigma)^\perp$ as

$$\mathrm{II}(X_p, Y_p) = (\overline{\nabla}_{X_p} Y)^\perp = \overline{\nabla}_{X_p} Y - \nabla_{X_p} Y \tag{2.3}$$

for vector fields X, Y tangent to Σ.

Given a local orthonormal field $\{E_i\}$ on Σ, the *mean curvature vector field* is then defined as

$$H = \frac{1}{k} \sum_{i=1}^{k} \mathrm{II}(E_i, E_i). \tag{2.4}$$

In general relativity, the second fundamental form and mean curvature vector field of a submanifold are sometimes defined as the negative of their usual definitions in differential geometry. Unless specifically stated otherwise, we will use the standard definitions above.

Given a C^2 function $f : M \to \mathbb{R}$, the Hessian of the restriction $u = f|_\Sigma : \Sigma \to \mathbb{R}$ is given by

$$\nabla^2 u(x,y) = \overline{\nabla}^2 f(x,y) + g(\mathrm{II}(x,y), \overline{\nabla} f). \tag{2.5}$$

This type of calculation is important when considering the behavior of the restrictions of convex or λ-convex functions to submanifolds.

We now review some basic causality theory.

Definition 2.1.6. A vector $v \in T_pM$ *timelike, null, or spacelike* if $g(v,v) < 0, g(v,v) = 0$, or $g(v,v) > 0$, respectively.

A piecewise C^1 curve γ in M will be called *timelike, null, or spacelike* if whenever it exists γ' is timelike, null, or spacelike, respectively, along γ.

Definition 2.1.7. A subspace $\sigma \subset T_pM$ is called *timelike* if $g|_\sigma$ is non-degenerate and indefinite, *spacelike* if $g|_\sigma$ is (positive or negative) definite, or *degenerate* otherwise. Likewise, a submanifold $\Sigma \subseteq M$ will be called timelike or spacelike if the tangent space at any point in N is a timelike or spacelike subspace, respectively.

Suppose M is a Lorentzian manifold. In a Lorentzian manifold we say a vector is *causal* if it is non-spacelike. Similarly, a curve is causal if at each point, its tangent vector is non-spacelike. For each $p \in M$ the set of all non-zero causal vectors in T_pM consists of two convex connected components, that may be called *hemicones*. A continuous choice of hemicone for all $p \in M$ is called a *time orientation* of M. Tangent vectors in the chosen hemicones and curves whose tangent vectors always lie in the chosen hemicones are called *future-directed*. A tangent vector v or a curve γ is called *past-directed* if $-v$ or $\gamma(-t)$ are future-directed, respectively. A Lorentzian manifold with a choice of time orientation is called a *space-time*.

Not all Lorentzian manifolds are time-orientable, but a Lorentzian manifold which is not time-orientable always admits a two-fold cover which is time-orientable [BEE96, Theorem 3.3].

A subset $U \subset M$ is called convex if any pair of points $p, q \in U$ are joined by a unique geodesic segment of (M, g) contained entirely in U denoted $[pq]$. As in Riemannian geometry, around every point in a semi-Riemannian manifold (in particular, every space-time) there are arbitrarily small convex neighborhoods that are diffeomorphic images under the exponential map. These are called *normal neighborhoods*.

We define the *energy* of a piecewise smooth curve $\gamma : [a, b] \to M$ with breaks at $a = t_0 < t_1 < \cdots < t_m = b$ as

$$E(\gamma) = \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} g(\gamma'(s), \gamma'(s))\, ds. \tag{2.6}$$

Proposition 2.1.8. *Let $U \subset M$ be a convex neighborhood. Then*

1. *If there is a timelike (resp. causal) curve in U from p to q, then $[pq]$ is timelike (resp. causal).*

2. *If $[pq]$ is timelike, then $E([pq]) \leq E(\alpha)$ for any causal piecewise smooth curve α in U with p and q as its endpoints, with equality if and only if α is a reparametrization of $[pq]$.*

Thus, causal geodesics are the local minimizers of the energy functional between causally connected points.

For a pair of points p, q in a space-time M, we write $p \ll q$ if there is a future-directed timelike curve from p to q and $p \leqslant q$ if there is a future-directed causal curve from p to q.

Definition 2.1.9. Suppose M is a space-time. We define the *chronological future* $I_M^+(p)$ and *chronological past* $I_M^-(p)$ of a point $p \in M$ as

$$I_M^+(p) = \{q \in M : p \ll q\} \quad \text{and} \quad I_M^-(p) = \{q \in M : q \ll p\}$$

and the *causal future* $J_M^+(p)$ and *causal past* $J_M^-(p)$ as

$$J_M^+(p) = \{q \in M : p \leqslant q\} \quad \text{and} \quad J_M^-(p) = \{q \in M : q \leqslant p\}.$$

For a subset $S \subset M$, we define

$$I_M^\pm(\Sigma) = \bigcup_{p \in \Sigma} I_M^\pm(p) \quad \text{and} \quad J_M^\pm(\Sigma) = \bigcup_{p \in \Sigma} J_M^\pm(p).$$

A set Σ is called *achronal* if no two points of Σ are causally related.

An open set U in a space-time M is called *causally convex* if no causal curve intersects U in a disconnected set. Given $p \in M$, the space-time M is called *strongly causal* at p if p has arbitrarily small causally convex neighborhoods, i.e. p has arbitrarily small convex neighborhoods so that no causal curve that leaves such a neighborhood ever returns. M is *strongly causal* if it is strongly causal at all points in M.

M is *globally hyperbolic* if it is strongly causal and $J_M^+(p) \cap J_M^-(q)$ is compact in M whenever $p \leqslant q$. The main results on geodesic connectedness in [CS08] apply to globally hyperbolic space-times with orthogonal splitting metrics satisfying analytic growth conditions (see Theorem 3.8.2). Much of our results on geodesic connectedness in Chapter 3 apply to strongly causal space-times or more generally null-disprisoning semi-Riemannian manifolds (see Definition 3.2.2).

A *Cauchy hypersurface* Σ is a subset of a space-time which is intersected by every inextendible causal curve exactly once. One can think of a Σ as a moment in time. Globally hyperbolic space-times are important because they admit Cauchy hypersurfaces and this allows for well-posedness of the Cauchy problem for the wave equation on a Lorentzian manifold [Ler52]. In fact, a space-time is globally hyperbolic if and only if it admits a Cauchy hypersurface [HE93, p. 211].

Finally, we briefly discuss the *Lorentzian distance function* $d : M \times M \to [0, \infty]$ for a space-time M. If $c : [0, 1] \to M$ is a piecewise smooth causal curve differentiable except at $0 = t_1 < t_2 < \cdots < t_k = 1$, then the length $L(c)$ of c is given by

$$L(c) = \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{-g(c'(t), c'(t))}dt.$$

If $p \ll q$, there are timelike curves of arbitrarily small length connecting p to q, so rather than taking the infimum to get the distance, we take the supremum over all such curves as the Lorentzian distance $d(p, q)$. In a convex neighborhood U, the future directed timelike geodesic in U from p to q will maximize the length function among all causal curves connecting p and q. For points q outside of $J_M^+(p)$, we define $d(p, q) = 0$. Similarly, one may define the Lorentzian distance $d_\Sigma : M \to [0, \infty]$ to a set Σ as $d_\Sigma(q) = \sup_{p \in \Sigma} d(p, q)$.

The results of Alías, Bessa and deLira in [ABL16] use Lorentzian distance functions and timelike curvature bounds to rule out weakly future-trapped submanifolds in subdomains of $J_M^+(p)$ and $J_M^+(\Sigma)$ where Σ is an achronal spacelike hypersurface. In Chapter 4, we discuss how the condition $\mathcal{R} \leq K$ may be used to extend these types of results beyond the causal future of a point.

2.2 Examples & Constructions

In this section, we introduce some basic examples and constructions of semi-Riemannian and Lorentzian manifolds.

Fix a pair of natural numbers k and n. The fundamental example of a semi-Riemannian manifold is semi-Euclidean space $\mathbb{E}_k^{n+k}$.

Example 2.2.1. *Semi-Euclidean space* $\mathbb{E}_k^{n+k}$ is $\mathbb{R}^{n+k}$ equipped with the metric

$$g = (dx^1)^2 + \cdots + (dx^n)^2 - (dx^{n+1})^2 - \cdots - (dx^{n+k})^2.$$

Semi-Euclidean space is flat and geodesics are simply line segments (parametrized linearly) of $\mathbb{R}^{n+k}$. When $k = 1$, semi-Euclidean space is called *Minkowski space* and can be endowed with the natural time-orientation in which ∂_{n+1} is a timelike vector field, making it a space-time.

Many of the semi-Riemannian manifolds (or space-times) we discuss in this thesis occur as submanifolds of semi-Euclidean space (or Minkowski space). The most important submanifold of Minkowski space for the purposes of comparison geometry is de Sitter space.

Example 2.2.2. For $n \geq 2$, *de Sitter space* $dS_1^n \subset \mathbb{E}_1^{n+1}$ is the hypersurface of points satisfying the equation

$$(x^1)^2 + \cdots + (x^n)^2 - (x^{n+1})^2 = 1$$

with the submanifold metric inherited from Minkowski space.

De Sitter space has constant sectional curvature $+1$ and is the Lorentzian analog to the sphere in Riemannian geometry. As in the sphere, the images of

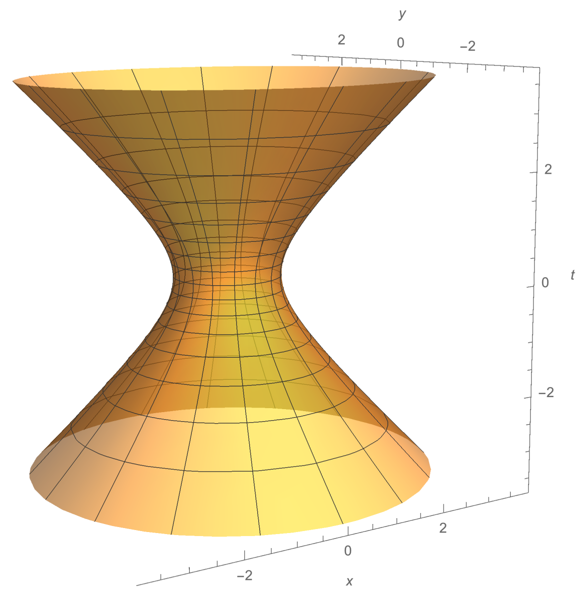

Figure 2.1: 2-dimensional de Sitter space in $\mathbb{E}^3_1$.

geodesics of de Sitter space are the connected components of intersections of dS^n_1 with 2-planes through the origin of $\mathbb{E}^{n+1}_1$, i.e. arcs of "great circles" in dS^n_1. Note that de Sitter space is *not* geodesically connected. For example in dS^2_1, the points $(\pm 2, 0, \sqrt{3})$ cannot be connected by a geodesic because such a geodesic would have to go through the "future pole" or "past pole" which do not exist as points in de Sitter space.

Example 2.2.3. For $n \geq 2$, let M be the hypersurface of $\mathbb{E}^{n+1}_2$ satisfying the equation

$$(x^1)^2 + \cdots + (x^{n-1})^2 - (x^n)^2 - (x^{n+1})^2 = -1.$$

When $n = 2$, $M = -dS^2_1$. Note that M is Lorentzian as a submanifold of $\mathbb{E}^{n+1}_2$ and contains closed timelike curves, an undesirable property from the perspective of physics. We resolve this by passing to the universal cover. *Anti-de Sitter space*, denoted by adS^n_1, is the universal cover of M with the metric inherited via the covering map.

Anti-de Sitter space has constant sectional curvature -1 and is the Lorentzian analog to hyperbolic space for Riemannian geometry. The images of geodesics of M (as defined above) are the connected components of intersections of two-planes through the origin of $\mathbb{E}^{n+1}_2$ with M (as defined above) and geodesics of adS^n_1 are lifts of such curves.

Two-dimensional de Sitter space is an example of a surface of revolution in $\mathbb{E}^3_1$ parametrized with the map $(s, \theta) \mapsto (\cosh s \cos\theta, \cosh s \sin\theta, \sinh s)$. In general, a timelike surface of revolution in $\mathbb{E}^3_1$ can be constructed using a unit speed timelike profile curve $\gamma : I \to \mathbb{E}^2_1 \subset \mathbb{E}^3_1$ by $\gamma(s) = (r(s), 0, \tau(s))$ with $r(s) > 0$ and rotating it about the x^3-axis to obtain a surface of revolution parametrized with the map

$$(s,\theta) \mapsto (r(s)\cos\theta, r(s)\sin\theta, \tau(s)).$$

The metric inherited from $\mathbb{E}^3_1$ in these coordinates is $-ds^2 + r(s)^2 d\theta^2$. This construction can be generalized using the notion of a warped product, originally introduced for Riemannian manifolds by Bishop and O'Neill in [BO69].

Definition 2.2.1. Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds and $f : B \to (0,\infty)$ a smooth positive function on B. The *warped product* denoted by $B \times_f F$ is the product manifold $B \times F$ with the semi-Riemannian metric $g = g_B + f^2 g_F$. B is called the *base* and F is called the *fiber*.

A *Generalized Robertson-Walker (GRW) space* is a warped product of the form $(-I) \times_f F$ where $I = (a,b) \subset \mathbb{R}$ is an interval and F is a Riemannian manifold.

Surfaces of revolution are examples of warped products. In dimension $n \geq 2$, de Sitter space dS^n_1 can be written as the GRW space $(-\mathbb{R}) \times_{\cosh} S^{n-1}$ where S^{n-1} is the $(n-1)$-dimensional sphere. Certain domains of anti-de Sitter space can be written as the GRW space $-(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} \mathbb{H}^{n-1}$ where $\mathbb{H}^{n-1}$ is $(n-1)$-dimensional hyperbolic space. The standard cosmological models of the large scale universe in General Relativity are GRW spaces of the form $(-I)\times_f F$ where F is a 3-dimensional Riemannian manifold of constant curvature [O'N83, HE93].

The following theorem of Alexander and Bishop establishes a large class of warped products satisfying $\mathcal{R} \geq K$ and $\mathcal{R} \leq K$.

Theorem 2.2.2. *[AB08, Proposition 7.1] Let B and F be Riemannian manifolds, $f : -B \to (0,\infty)$ be a smooth function, ∇f denote the gradient of f on $-B$, $\langle\cdot,\cdot\rangle$ be the metric on $-B$, and $\mathcal{K}_F$ the sectional curvature function on F. Then $(-B) \times_f F$ is a semi-Riemannian manifold satisfying $\mathcal{R} \geq K$ ($\mathcal{R} \leq K$) if and only if the following three conditions hold:*

1. *f satisfies $\nabla^2 f(x,x) \leq (\geq) - Kf\langle x, x\rangle$,*
2. *$\dim B = 1$ or B has sectional curvature $\leq (\geq) - K$,*
3. *$\dim F = 1$, or for all points $(p,\overline{p}) \in (-B) \times_f F$ and 2-planes $\Pi_{\overline{p}}$ tangent to F, $\mathcal{K}_F(\Pi_{\overline{p}}) \geq (\leq) K f(p)^2 + \langle \nabla f_p, \nabla f_p\rangle$.*

We can apply the above theorem to GRW spaces to obtain the following corollary.

Corollary 2.2.3. *[AB08] A GRW space $M = (-I) \times_f F$ satisfies $\mathcal{R} \leq K$ if and only if $f : I \to (0,\infty)$ satisfies $f'' \geq -Kf$, and F is either 1-dimensional or has sectional curvature $\leq C$ where $C = \inf_{t\in I}(Kf(t)^2 - f'(t)^2)$.*

(For $\mathcal{R} \geq K$, reverse the inequalities and substitute sup *for* inf*.)*

Another class of space-times we will discuss is *timelike convex hypersurfaces* in Minkowski space. A convex hypersurface is the connected boundary of a convex set with nonempty interior. When a convex hypersurface is a timelike submanifold of Minkowski space, it inherits a Lorentzian submanifold metric and a time-orientation from Minkowski space. Locally, a timelike convex hypersurface M is the graph of a convex function on the tangent plane to a point in M. This can be used to show that timelike convex hypersurfaces satisfy $\mathcal{R} \geq 0$ (see Proposition 3.6.15). To produce a convex function on a timelike convex hypersurface M for the purposes of applying our main theorem on geodesic connectedness, we restrict a carefully chosen affine function on Minkowski space to M (see Theorem 3.6.12).

The following two examples will be important in Chapter 3.

Example 2.2.4. Let C^{n-1} be a smoothly capped cylinder embedded as a convex hypersurface of the copy of Euclidean space $\{x^{n+1} = 0\}$ in $\mathbb{E}_1^{n+1}$. Let the cylindrical part of C^{n-1} be given by

$$x^1 \geq 1, \quad |x^2|^2 + \cdots + |x^n|^2 = 1, \quad x^{n+1} = 0, \tag{2.7}$$

and let the cap lie in the region $0 \leq x^1 \leq 1$. Set

$$M = \{p + (\sqrt{1+t^2}, 0, \ldots, 0, t) : t \in \mathbb{R}, p \in C^{n-1}\}.$$

Since M contains, through each if its points, a translate of the timelike convex curve $\gamma(t) = (\sqrt{1+t^2}, 0, \ldots, 0, t)$ in the x^1x^{n+1}-plane, M is timelike. To see that M is a convex hypersurface, one can verify that the normal vector to M at the point $p + (\sqrt{1+t^2}, 0, \ldots, 0, t)$ can be given by $N_p + \frac{t}{\sqrt{1+t^2}}\langle N_p, \partial_1\rangle \partial_{n+1}$, where N_p is the normal vector to C^{n-1} at p, and that M is contained in a half-space orthogonal to its normal vector at each point.

For $n > 2$, M contains closed spacelike geodesics in the $(n-2)$-dimensional sphere $|x^2|^2 + \cdots + |x^n|^2 = 1, x^{n+1} = 0, x^1 = 2$. In Chapter 3, the method of proof of our main theorem on geodesic connectedness for a semi-Riemannian manifold M requires that M be *disprisoning*, i.e. for any inextendible geodesic of M, neither of its ends may be contained in a compact set. We do not know if this condition is necessary for geodesic connectedness, but it is necessary for our method of proof.

Example 2.2.5. Let $C = \{(x, y, z, t) \in \mathbb{E}_1^4 : z = (x-t)^2 + y^2\}$. C is obtained by taking a paraboloid $z = x^2 + y^2$ and translating it at the speed of light in the x-direction. C has the interesting property that it is a timelike submanifold of $\mathbb{E}_1^4$, is ruled by parallel null lines in the direction of the vector $(1, 0, 0, 1)$, but it does not contain any spacelike or timelike lines of $\mathbb{E}_1^4$. If a timelike convex hypersurface M of $\mathbb{E}_1^{n+1}$ contains a spacelike or timelike line L, M may be decomposed as a product $L \times M_0$, where M_0 is a convex hypersurface of the orthogonal complement of L in $\mathbb{E}_1^{n+1}$. In Section 3.6, the method of proof of our main theorem on geodesic connectedness of timelike convex hypersurfaces of Minkowski space fails when, after splitting off a maximal number of timelike and spacelike lines, the remaining convex hypersurface M_0 is ruled by null lines.

However, in the situation where M is ruled by parallel null lines, other recent methods developed by Bartolo, Candela, and Flores in [BCF14] can be applied to prove that M is geodesically connected. See Section 3.9 for a full discussion of this topic.

Chapter 3

Convex functions and geodesic connectedness

3.1 Introduction

This chapter explores the relation between geometric convexity, and geodesic connectedness of space-times and semi-Riemannian manifolds. We consider geodesics of all causal types, since they form the scaffolding for the global topological and geometric structure of the space. With the exception of Section 3.9, the material of this chapter is from [AK16].

According to Gibbons and Ishibashi [GI01]: "Convexity and convex functions play an important role in theoretical physics. For example, Gibbs's approach to thermodynamics [Gibbs] is based on the idea that the free energy should be a convex function. A closely related concept is that of a convex cone which also has numerous applications to physics. Perhaps the most familiar example is the light cone of Minkowski space-time. Equally important is the convex cone of mixed states of density matrices in quantum mechanics. Convexity and convex functions also have important applications to geometry, including Riemannian geometry [Ud94]. It is surprising therefore that, to our knowledge, that [sic] techniques making use of convexity and convex functions have played no great role in General Relativity."

Sufficient conditions for geodesic connectedness of Lorentzian manifolds are given by an early theorem of Uhlenbeck [Uh75, Theorem 5.3], and by [BEE96, Theorem 11.25]. However, these theorems concern spaces with no conjugate points, whereas the spaces we consider may have conjugate points along geodesics of all causal types.

Geodesic connectedness was studied via an infinite-dimensional variational theory introduced by Benci, Fortunato, Giannoni and Masiello at the end of the 1980s (see [M94, M06]). For Lorentzian manifolds carrying a timelike or null Killing field, geodesic connectedness has only recently become well understood [CFS08, BCF14]. It is also known to hold for globally hyperbolic space-times carrying time-dependent orthogonal splittings satisfying certain conditions, as summarized in Theorem 3.8.2. See the informative survey of geodesics in semi-Riemannian manifolds by Candela and Sanchez [CS08].

Globally hyperbolic manifolds always have orthogonal splittings [BS05], but there may be none satisfying the conditions just mentioned, e.g. de Sitter space, which is not geodesically connected. Or there might exist splittings that satisfy the conditions, but no known way to determine their existence.

According to [CS08], "it should be interesting to obtain a result similar to that one also under weaker assumptions on the metric or under intrinsic hypotheses more related to the geometry of the manifold."

Uhlenbeck considers orthogonal splittings satisfying a metric growth condition, and also calls for a more geometric approach, observing that the growth condition is "not very satisfactory since it depends on the splitting [which] may be changed in drastically different ways ... it is to be hoped that a similar condition that does not depend on coordinates may be found" [Uh75, p. 75].

Using convex functions, we give geometric/topological proofs of geodesic connectedness for classes of space-times to which known methods do not apply. Our theorems concern space-times that are strongly causal or, more generally, null-disprisoning (see Definition 3.2.2); or else timelike convex hypersurfaces, which while globally hyperbolic, typically do not have natural orthogonal splittings that satisfy all the conditions in [CS08] (see Section 3.8), nor do we know how to determine if any splitting that satisfies the conditions exists.

Convex hypersurfaces of $\mathbb{E}^{n+1}$ are Riemannian manifolds of sectional curvature $\mathrm{Sec} \geq 0$, and their properties reflect those of general Riemannian manifolds of $\mathrm{Sec} \geq 0$. Timelike convex hypersurfaces of $\mathbb{E}^{n+1}_1$ satisfy $\mathcal{R} \geq 0$. Thus our motivation for studying timelike convex hypersurfaces is two-fold: They are space-times to which topological/geometric arguments readily apply, and in particular they carry convex functions. And as in the Riemannian case, they should be a guide to properties of more general space-times of $\mathcal{R} \geq 0$ (for some properties of $\mathcal{R} \geq 0$, see Remark 3.2.4 below).

3.2 Results

Recall that by a *convex (strictly convex) function* on a semi-Riemannian manifold, we mean a smooth real-valued function whose restriction to every geodesic has nonnegative (positive) second derivative. Equivalently, f is convex (strictly convex) if and only if the Hessian $\nabla^2 f$ is positive semidefinite (positive definite).

Remark 3.2.1. This chapter demonstrates the importance of these classically convex functions (equivalently, taking the negative, concave functions) in studying certain space-times that satisfy the curvature condition $\mathcal{R} \geq 0$. On Riemannian spaces with sectional curvature ≥ 0, convex functions arise naturally (see [CG72, Pts97]).

On Riemannian spaces with sectional curvature ≤ 0, convex functions again arise naturally (see [BO69] and examples in [Ud94, Ch.4]). In Chapter 4, we explore a close relationship between space-times satisfying $\mathcal{R} \leq K$ and space-time convex functions.

In an early and influential consideration of geodesic connectedness of Riemannian manifolds, Gordon proved that if a connected Riemannian manifold M supports a proper, nonnegative convex function, then M is geodesically connected [Go74]. Gordon's proof depends on the fact that complete Riemannian manifolds are geodesically connected, and does not extend to the Lorentz setting where geodesic connectedness is not a consequence of any completeness hypothesis.

We prove the following semi-Riemannian version of Gordon's theorem.

Definition 3.2.2. A semi-Riemannian manifold M is called *disprisoning* if for every inextendible geodesic $\gamma : (a, b) \to M$, neither end lies in a compact set. M is called *null-disprisoning* if for every inextendible null geodesic, neither end lies in a compact set.

Note that strongly causal, in particular globally hyperbolic, space-times are null-disprisoning [BEE96, Proposition 3.13].

Theorem 3.2.3. *Let M be a null-disprisoning semi-Riemannian manifold. Suppose M supports a proper, nonnegative convex function $f : M \to \mathbb{R}$ whose critical set is a minimum point. If there is no non-constant complete geodesic on which f is constant (for example, if f is strictly convex), then M is geodesically connected.*

Remark 3.2.4. In Riemannian comparison theory, the existence of proper non-negative convex functions plays a fundamental role. A complete Riemannian manifold of nonnegative sectional curvature always carries such a function, obtained by taking the negative of the infimum of all Busemann functions of rays based at a point. The Soul Theorem of Meyer-Cheeger-Gromoll is a consequence (see [Pts97, §11.4]).

We have already mentioned that timelike convex hypersurfaces M of $\mathbb{E}^{n+1}_1$ satisfy $\mathcal{R} \geq 0$ (namely, timelike sectional curvatures ≤ 0 and spacelike ones ≥ 0); see Proposition 3.6.15. Moreover, they support proper convex functions (Theorem 3.6.12). We expect timelike convex hypersurfaces M to indicate properties of more general space-times of $\mathcal{R} \geq 0$. Thus we come to the question: do space-times with $\mathcal{R}$ bounds have a rich structure analogous to Riemannian comparison theory? We mention some affirmative indicators:

1. Andersson and Howard proved "gap" rigidity theorems for $\mathcal{R} \geq 0$ and $\mathcal{R} \leq 0$ of the type first proved for Riemannian manifolds of $\mathrm{Sec} \geq 0$ and $\mathrm{Sec} \leq 0$ by Greene and Wu [GW82] and Gromov [BGS85] respectively [AH98].

2. Using the Penrose trapped surface theorem, Mukuno recently proved an analog of Myers' Theorem for null-geodesically complete Lorentzian manifolds M with metric of the form $-dt^2 + g_{\mathrm{Riem}}(t)$ and compact second factor [M14]. Namely, if M satisfies $\mathcal{R} \geq \kappa$ for $\kappa > 0$, then M has finite fundamental group.

3. In Riemannian manifolds, sectional curvature bounds are characterized by local distance comparisons. In [AB08], an analogous theorem is shown to hold in semi-Riemannian manifolds having an $\mathcal{R}$ bound.

Definition 3.2.5. A *convex body* in $\mathbb{E}_k^{n+k}$ is a closed convex set (not assumed compact) with nonempty interior. A *convex hypersurface* of $\mathbb{E}_k^{n+k}$ is a connected smooth manifold that is smoothly embedded as the boundary of a convex body.

We prove the following geodesic connectedness theorem for timelike convex hypersurfaces of Minkowski space $\mathbb{E}_1^{n+1}$.

Theorem 3.2.6. *Let M be a timelike convex hypersurface of $\mathbb{E}_1^{n+1}$. Suppose that after splitting off a semi-Euclidean factor of maximal dimension,*

1. *M does not contain an isometrically immersed Euclidean half-plane $\{(x, y) : x \in \mathbb{R}, y \geq 0\}$ where the generating half-lines $x = \text{constant}$ are carried to parallel half-lines of $\mathbb{E}_1^{n+1}$,*
2. *M is not ruled by parallel null lines of $\mathbb{E}_1^{n+1}$.*

Then M is geodesically connected.

In particular, any timelike strictly convex hypersurface is geodesically connected.

Remark 3.2.7. We do not know if condition (1) of Theorem 3.2.6 is necessary. Condition (2) can be removed using recent results of Bartolo, Candela, and Flores in [BCF14]. See Section 3.9.

An example of a timelike strictly convex surface is examined and illustrated in Section 3.8.

Example 3.2.1. Recall Example 2.2.4 obtained by taking a convex smoothly capped half-cylinder $\{x^1 \geq 1, |x^2|^2 + \cdots |x^n|^2 = 1\}$ in the hyperplane $\{x^{n+1} = 0\}$ of Minkowski space and translating it along the curve $\gamma(t) = (\sqrt{1+t^2}, 0, \cdots, 0, t)$.

For $n > 2$, M contains an isometrically immersed image of the Euclidean half-plane as described in Theorem 3.2.6(1): namely, for $\{(x, y) : x \in \mathbb{R}, y \geq 0\}$, let the images of the half-lines $x = \text{constant}$ be the generating half-lines of the cylinder (2.7) and let the base curve $y = 0$ cover a great circle in the $(n-2)$-sphere given by $x^1 = 1$, $|x^2|^2 + \ldots |x^n|^2 = 1$. Thus M satisfies the hypotheses of Theorem 3.2.6 when $n = 2$, but fails to satisfy condition (1) when $n > 2$.

The following corollary of Theorem 3.2.3 generates yet more geodesically connected space-times:

Definition 3.2.8. A smooth function $f : M \to \mathbb{R}$ on a Lorentzian manifold M will be called *Lorentzian* if the graph of f in $M \times \mathbb{R}$ is a timelike submanifold, i.e. $\langle \nabla f, \nabla f \rangle > -1$.

Corollary 3.2.9. *Let M be a connected, strongly causal space-time, and $f : M \to \mathbb{R}$ be a proper, nonnegative strictly convex Lorentzian function. Then the graph of f in $M \times \mathbb{R}$ is geodesically connected.*

Geodesic connectedness can also be diagnosed using not-necessarily-convex functions. Specifically, in Theorem 3.7.1 we give a criterion for the levels of a function on a semi-Riemannian manifold to be the levels of a convex function.

The criterion is related to Fenchel's criterion for deciding if a function defined on an affine space and having convex level sets can be reparametrized as a convex function [F53]. Theorems 3.7.1 and 3.2.3 allow us to extend our class of geodesically connected spaces.

Here is a special case of these theorems. The negativity of the expression μ measures how badly the function u fails to be convex.

Theorem 3.2.10. *Suppose $u : M \to \mathbb{R}$ is a proper smooth nonnegative function on a connected semi-Riemannian manifold M, where the critical set of u is a minimum point p_0, say $u(p_0) = 0$. For $a \in \operatorname{range} u - \{0\}$, suppose the level sets L_a are infinitesimally strictly convex, i.e. $\nabla^2 u(\mathbf{x}, \mathbf{x}) > 0$ if $\mathbf{x} \in T_p L_a$.*

Let N be a vector field on $M - \{p_0\}$ satisfying $Nu > 0$. For $a \in \operatorname{range} u - \{0\}$, set

$$\mu(a) = \inf\big\{\big[\nabla^2 u(\mathbf{x}, \mathbf{x})\nabla^2 u(N_p, N_p) - \big(\nabla^2 u(\mathbf{x}, N_p)\big)^2\big]/(N_p u)^2 \nabla^2 u(\mathbf{x}, \mathbf{x}) : \\ p \in L_a, \mathbf{x} \in T_p L_a\big\}.$$

If μ is bounded below by a continuous function $h : \operatorname{range} u - \{0\} \to \mathbb{R}$ that extends continuously to 0, then:

1. *There is a smooth function $\rho : [0, \infty) \to [0, \infty)$ such that $\rho' \geq 1$ and $f = \rho \circ u$ is a proper strictly convex function.*
2. *If M is null-disprisoning, then M is geodesically connected.*
3. *If M is a strongly causal space-time and u is Lorentzian, then the graph in $M \times \mathbb{R}$ of u is geodesically connected.*

Remark 3.2.11. In applications, it is often possible to verify the hypothesis on μ by showing that μ is continuous and finite.

As an application, we construct a large class of non-convex Lorentzian hypersurfaces in $\mathbb{E}^{n+1}_1$ that are geodesically connected (Corollary 3.7.5).

In Section 3.8, we give an example that is geodesically connected by Theorem 3.2.6, yet does not appear to carry orthogonal splittings that satisfy the growth conditions required by the main theorem of [CS08].

3.3 Convex functions and geodesic connectedness

Definition 3.3.1. For a semi-Riemannian manifold M, SM will denote the unit tangent bundle for some Riemannian metric g_{Riem} on M. When we write SM or S_pM, it means we have made a choice of g_{Riem}.

Throughout this section, for a given convex function $f : M \to \mathbb{R}$ and any non-critical value a of f, we denote the level sets by $L_a = \{p \in M : f(p) = a\}$ and the sublevel sets by $M_a = \{p \in M : f(p) \leq a\}$.

Lemma 3.3.2. *Let M be a null-disprisoning semi-Riemannian manifold, and $f : M \to \mathbb{R}$ be a nonnegative proper convex function. Suppose the critical set C of f is connected, so C is the minimum set, say $f|_C = 0$. Then one of these two statements holds:*

1. *M is disprisoning,*
2. *There is a complete non-constant geodesic γ such that $f \circ \gamma$ is constant.*

Proof. Since M is null-disprisoning, M is non-compact. Since f is proper, the values of f are unbounded.

Suppose M is not disprisoning. Then there exists $p \in M_a$, and a right-sidedly maximally extended geodesic α with left-hand endpoint $\alpha(0) = p$, such that α does not leave M_a.

Suppose α is defined on $[0, b)$, $b \leq \infty$. Consider an increasing sequence $t_i \to b$, and a sequence $(\alpha(t_i), \mathbf{v}_i) \in SM$ where $\mathbf{v}_i$ has the same direction as $\alpha'(t_i)$. Since the $\alpha(t_i)$ lie in a compact set, we may assume $(\alpha(t_i), \mathbf{v}_i) \to (q, \mathbf{v}) \in SM$.

Claim 1. $\mathbf{v}$ is not null.

Suppose $\mathbf{v}$ is null. Then the maximally extended geodesic with left-hand endpoint q and initial condition $(q, \mathbf{v})$ leaves M_a by hypothesis. By continuous dependence of geodesics on initial conditions, $\alpha|_{[t_i,b)}$ leaves M_a for i sufficiently large. This contradiction proves the claim.

Claim 2. α is defined on $[0, \infty)$.

Suppose $b < \infty$. Since $\mathbf{v}$ is not null, as i increases the $(\alpha(t_i), \mathbf{v}_i)$ lie in compact neighborhoods of $(q, \mathbf{v})$ whose intersection is $(q, \mathbf{v})$. Then the existence of normal coordinate neighborhoods guarantees that α is the only geodesic that approaches q with bounded affine parameter. Therefore α extends to $\alpha(b) = q$, contradicting maximality.

Claim 3. There is a complete non-constant geodesic γ such that $f \circ \gamma$ is constant.

Since $f \circ \alpha$ is convex and bounded above and below, $f \circ \alpha$ is non-increasing and

$$\lim_{t \to \infty} (f \circ \alpha)(t) = c$$

for some $c \leq a$.

Choose a sequence $\varepsilon_i \to 0^+$. Let the sequence t_i as above increase to ∞ so that $c \leq f \circ \alpha_i \leq c + \varepsilon_i$, where $\alpha_i : [0, \infty) \to M$ is the geodesic with $\alpha_i(0) = \alpha(t_i)$, $\alpha_i'(0) = \mathbf{v}_i$. By continuous dependence of geodesics on initial conditions, the geodesic $\gamma : [0, \infty) \to M$ with $\gamma(0) = q$, $\gamma'(0) = \mathbf{v}$ is defined on $[0, \infty)$ and satisfies $f \circ \gamma = c$. Moreover, we can choose $t_i - i > t_{i-1}$ for each i. By claim 1, dt/ds_i is bounded above, where t is the parameter of α and s_i is the parameter of α_i. It follows that γ extends to $(-\infty, \infty)$ with $f \circ \gamma = c$.

This completes the proof of the Lemma. □

Lemma 3.3.3. *Let $f : M \to \mathbb{R}$ be a convex function on a semi-Riemannian manifold M. Suppose a geodesic γ satisfies $\gamma|_{[t_0-\varepsilon, t_0)} \subset M_a - L_a$, $\gamma(t_0) \in L_a$, where a is a non-critical value of f. Then γ intersects L_a transversely at $\gamma(t_0)$.*

Proof. Since $f \circ \gamma$ is convex and $(f \circ \gamma)(t_0 - \varepsilon) < (f \circ \gamma)(t_0)$, then $(f \circ \gamma)'(t_0) > 0$. Hence $\gamma'(t_0)$ is not tangent to the level set L_a. □

Lemma 3.3.4. *Let M be a disprisoning semi-Riemannian manifold and $f : M \to \mathbb{R}$ be a nonnegative and non-constant proper convex function. Suppose the critical set C of f is connected, so C is the minimum set, say $f|_C = 0$. For $a > 0$ and $p \in M_a - L_a$, consider the map $\psi_{(p,a)} : S_pM \to L_a$, where $\psi_{(p,a)}(\mathbf{v})$ is the first point at which the geodesic of M with initial velocity $\mathbf{v}$ leaves M_a. Then:*

1. *$\psi_{(p,a)}$ is continous;*
2. *$\psi_{(p,a)}$ varies continously with $p \in M_a - L_a$;*
3. *L_a is connected.*

Proof.

Claim 1. M_a is connected.

If $[a, b]$ contains no critical value of f then M_a is a deformation retract of M_b, where for any choice of Riemannian metric g_{Riem} on M, the retraction map may be taken along downward gradient curves of f reparametrized by values of f. Thus for any $a > 0$, M_a is a deformation retract of M. By properness of f and connectedness of C, M is connected. Since M is connected, M_a is connected.

Claim 2. The level sets L_a, $a > 0$, are continuously diffeomorphic. Specifically, fix $\hat{a} > 0$. Then there is a diffeomorphism $\varphi : M - C \to (0, \infty) \times L_{\hat{a}}$ such that $\varphi|_{L_a}$ is a diffeomorphism onto $\{a\} \times L_{\hat{a}}$.

By disprisonment and properness, M is non-compact and f is unbounded. For any choice of Riemannian metric g_{Riem} on M, the map φ is given by the downward gradient curves of f, reparametrized by the value of f.

Claim 3. $M_a - L_a$ is connected.

Immediate from claims 1 and 2.

(1) and (2) in our lemma statement are consequences of Lemma 3.3.3, which implies that for geodesics whose initial directions in S_pM converge to that of γ, the parameter value of first departure from M_a also converges to that of γ.

To prove (3), suppose a geodesic γ from $p \in M_a - L_a$ first leaves M_a by intersecting the component L'_a of L_a. By (1), the directions of geodesics that first leave M_a by intersecting L'_a form a nonempty open and closed subset of S_pM. Thus every geodesic from p first leaves M_a by intersecting L'_a.

By (2), the points $p \in M_a - L_a$ from which geodesics from p first leave M_a by intersecting L'_a form an open and closed subset of $M_a - L_a$, hence all of $M_a - L_a$ by claim 3.

There can be no component $L''_a \neq L'_a$ of L_a. Indeed, a point of L''_a would have a normal coordinate neighborhood U in M such that $L_a \cap U$ lies in $L''_a \cap U$. Hence there would be geodesics from points in $M_a - L_a$ that first leave M_a by intersecting L''_a, a contradiction. □

The following technical lemma will be used to prove our theorems on geodesic connectedness, in particular Theorems 3.2.3 and 3.2.6. In both cases, the conditions on C will be easily verified.

Lemma 3.3.5. *Let M be a disprisoning semi-Riemannian manifold and $f : M \to \mathbb{R}$ be a nonnegative proper convex function. Suppose any two points of the critical set C of f are joined by a geodesic of M, and C has an oriented neighborhood U in M. Then M is oriented.*

Suppose further that for some non-critical value a, there is $p \in M_a - L_a$ such that the map $\psi_{(p,a)} : S_pM \to L_a$ has nonzero degree, where $\psi_{(p,a)}(\mathbf{v})$ is the first point at which the geodesic of M with initial velocity $\mathbf{v}$ leaves M_a. Then M is geodesically connected.

Proof. By convexity of f, critical points of f are local minima, C is geodesically connected, and C is the minimum set of f, say $f|_C = 0$.

Claim 1. M and the level sets L_a, $a > 0$, have an orientation determined by the given oriented neighborhood U of C.

Downward gradient flow of f in g_{Riem} carries a coordinate neighborhood of each point in M diffeomorphically into U. Hence coordinates on M may be chosen so that all transition functions have positive determinant. This orientation of M induces an orientation on each L_a by requiring the coordinate basis of T_pL_a, followed by the gradient vector in g_{Riem} of f at p, to be a positively oriented basis of T_pM.

Claim 2. For any $a > 0$, the degree of $\psi_{(p,a)}$ is constant for all $p \in M_a - L_a$.

The level set L_a is compact by properness of f, connected by Lemma 3.3.4, and oriented by claim 1. Thus degree of $\psi_{(p,a)}$ is defined.

By claim 3 of Lemma 3.3.4, two points of $M_a - L_a$ are joined by a path $\alpha : [0,1] \to M_a - L_a$. Parallel translation along α with respect to g_{Riem} identifies the pull-back bundle $\alpha^*(SM)$ of SM along α homeomorphically with $S^{n-1} \times [0,1]$. By disprisonment and Lemma 3.3.4, the maps $\psi_{(\alpha(t),a)} : S_{\alpha(t)}M \to L_a$ determine a one-parameter family of continuous maps $S^{n-1} \to L_a$. Since these maps vary continuously in t, their degree is constant.

Claim 3. For any $a > 0$ and every $p \in M_a - L_a$, there is a geodesic from p to every point of L_a.

By claim 2 of Lemma 3.3.4, the map

$$(\varphi|_{L_a}) \circ \psi_{(p,a)} : S_pM \to \{a\} \times L_{\hat{a}}$$

has constant degree for all $p \in M_a - L_a$. Moreover, this map varies continously in a, and so has constant degree for all $a > 0$ and all $p \in M_a - L_a$. Then the claim follows from our degree hypothesis.

Claim 4. For any $a > 0$ and every $p \in M_a$, there is a geodesic from p to every point of L_a.

By claim 3, it suffices to show that any two distinct points $p, q \in L_a$ are joined by a geodesic. In particular, for $p_i \to p$, $p_i \in M_a - L_a$, there is a geodesic $\gamma_i : [0, c_i] \to M_a$ satisfying $\gamma_i(0) = p_i$, $\gamma_i(c_i) = q$, $(p_i, \gamma_i'(0)) \in SM$. Since M_a is compact, we may assume the sequence $(p_i, \gamma_i'(0))$ converges in SM to $(p, \mathbf{v}) \in SM$, $\mathbf{v} \neq 0$. We may also assume $c_i \to c \in (0, \infty]$.

Let γ be the maximally extended geodesic with $\gamma(0) = p$, $\gamma'(0) = \mathbf{v}$. If $c < \infty$, then $\gamma|_{(0,c]}$ is defined and joins p to q. If $c = \infty$, then γ is defined on $[0, \infty)$ and does not leave M_a, a contradiction since M is disprisoning.

Claim 5. For every $p \in M_a$, there is a geodesic from p to every $q \in M_a$.

If $p, q \in C$, the claim is true by hypothesis. Otherwise, set

$$a = \max\{f(p), f(q)\}.$$

Then claim 5 follows from claims 3 and 4.

Hence the lemma. $\square$

Theorem 3.3.6 (Theorem 3.2.3)**.** *Let M be a null-disprisoning semi-Riemannian manifold. Suppose M supports a proper, nonnegative convex function f whose critical set is a point.*

If there is no non-constant complete geodesic on which f is constant (for example, if f is strictly convex), then M is geodesically connected.

Proof. Since the critical set is $C = \{p\}$, C has an orientable neighborhood. By Lemmas 3.3.2 and 3.3.5, it suffices now to observe that if a is sufficiently small, $\psi_{(p,a)}$ maps S_pM diffeomorphically onto L_a and hence has degree one. $\square$

3.4 Graphs and geodesic connectedness

In this section, we prove Corollary 3.2.9 on geodesic connectedness of graphs of strictly convex functions. Let us briefly review some basic Lorentzian terminology.

Recall that for two points $p, q \in M$ we write $p \leqslant q$ if $p = q$ or if there is a piecewise smooth curve with future-pointing (possibly one-sided) tangent vectors from p to q. The *causal future* of $p \in M$ is $J^+_M(p) = \{q \in M : p \leqslant q\}$ and the *causal past* is $J^-_M(p) = \{q \in M : q \leqslant p\}$.

An open neighborhood in a space-time M is *causally convex* if every piecewise smooth curve with future-pointing tangent vectors intersects it in a connected set. Recall that a space-time is *strongly causal* if every point has arbitrarily small causally convex neighborhoods. A strongly causal space-time is *globally hyperbolic* if $J^+_M(p) \cap J^-_M(q)$ is compact for all $p, q \in M$.

Lemma 3.4.1. *Suppose M is a Lorentzian manifold and $u : M \to \mathbb{R}$ is a Lorentzian function. Let $\Gamma(u)$ be the graph of u in $M \times \mathbb{R}$. Let the function $f : \Gamma(u) \to \mathbb{R}$ be the lift of u, defined by $f(p, u(p)) = u(p)$. Then for any vectors $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in T_{(p,u(p))}\Gamma(u)$, with corresponding vectors $\mathbf{x}, \mathbf{y} \in T_pM$ obtained by projection onto M,*

$$\nabla^2 f\,(\overline{\mathbf{x}}, \overline{\mathbf{y}}) = \frac{\nabla^2 u(\mathbf{x}, \mathbf{y})}{1 + \langle (\nabla u)_p, (\nabla u)_p \rangle}. \tag{3.1}$$

Proof. Suppose $\gamma(t) = (\alpha(t), u(\alpha(t)))$ is a geodesic of $\Gamma(u)$. Then the second covariant derivatives satisfy

$$\gamma''(t) = \alpha''(t) + (u \circ \alpha)''(t)\partial_y|_{(u\circ\alpha)(t)}$$

where ∂_y is the standard coordinate vector field on the second factor of $M \times \mathbb{R}$.

Any vector field Y on $\Gamma(u)$ can be written as $Y = X + (Xu)\partial_y = X + \langle \nabla u, X\rangle \partial_y$ where X is a vector field on M. In order for γ to be a geodesic, $\gamma''(t)$ must be orthogonal to $\Gamma(u)$ in $M \times \mathbb{R}$. Thus

$$\langle \gamma''(t), Y_{\gamma(t)}\rangle = \langle \alpha''(t), X_{\alpha(t)}\rangle + (u \circ \alpha)''(t)\langle (\nabla u)_{\alpha(t)}, X_{\alpha(t)}\rangle$$

$$= \langle \alpha''(t) + (u \circ \alpha)''(t)(\nabla u)_{\alpha(t)}, X_{\alpha(t)}\rangle = 0$$

for any vector field X on M, so

$$\alpha''(t) + (u \circ \alpha)''(t)(\nabla u)_{\alpha(t)} = \mathbf{0}. \tag{3.2}$$

Therefore

$$(u \circ \alpha)''(t) = \nabla^2 u(\alpha'(t), \alpha'(t)) + \langle (\nabla u)_{\alpha(t)}, \alpha''(t)\rangle$$

$$= \nabla^2 u(\alpha'(t), \alpha'(t)) - (u \circ \alpha)''(t)\langle (\nabla u)_{\alpha(t)}, (\nabla u)_{\alpha(t)}\rangle.$$

Moreover,

$$\nabla^2 f(\gamma'(t),\gamma'(t)) = (f\circ\gamma)''(t) = (u\circ\alpha)''(t) = \frac{\nabla^2 u(\alpha'(t),\alpha'(t))}{1+\langle(\nabla u)_{\alpha(t)},(\nabla u)_{\alpha(t)}\rangle}.$$

Since this holds for any geodesic and $\alpha'(t)$ is the projection of $\gamma'(t)$ to $T_{\alpha(t)}M$, we conclude that for any tangent vector $\overline{\mathbf{x}} \in T_{(p,u(p))}\Gamma(u)$,

$$\nabla^2 f\left(\overline{\mathbf{x}},\overline{\mathbf{x}}\right) = \frac{\nabla^2 u(\mathbf{x},\mathbf{x})}{1+\langle(\nabla u)_p,(\nabla u)_p\rangle}$$

where $\mathbf{x}$ is the projection of $\overline{\mathbf{x}}$ onto T_pM. Equation (3.1) follows since symmetric bilinear forms on vector spaces are determined by their corresponding quadratic forms. ☐

Lemma 3.4.2. *If M is a strongly causal space-time and H is a Riemannian manifold, then any immersed timelike submanifold E of $M \times H$ is a strongly causal space-time in the induced Lorentzian metric.*

Proof. By [BEE96, Lemma 3.54 and Proposition 3.62], $M \times H$ is a strongly causal space-time since M is a strongly causal space-time. Since the timelike tangent vectors to E form the intersection of TE with the timelike vectors in the pull-back of $T(M \times H)$, E inherits a time orientation from $M \times H$.

Suppose E is not strongly causal. For $p \in E$, every sufficiently small neighborhood $\widetilde{U}$ of p in $M \times H$ lies in a coordinate neighborhood whose intersection with E is a coordinate slice. Moreover, there is a piecewise smooth curve γ with future-pointing tangent vectors in the component U' of $E \cap \widetilde{U}$ containing p, such that γ intersects U' in a disconnected set. But then γ intersects $\widetilde{U}$ in a disconnected set. This contradiction shows E inherits strong causality from $M \times H$. ☐

Corollary 3.4.3 (Corollary 3.2.9)**.** *If $u : M \to \mathbb{R}$ is a proper, nonnegative, strictly convex Lorentzian function defined on a connected strongly causal space-time M, then the graph $\Gamma(u)$ of u is geodesically connected.*

Proof. Since the graph $\Gamma(u)$ is a timelike submanifold of $M \times \mathbb{R}$, then $\Gamma(u)$ is a strongly causal space-time by Lemma 3.4.2. Therefore $\Gamma(u)$ is null-disprisoning [BEE96, Proposition 3.13].

Let $f : \Gamma(u) \to \mathbb{R}$ be the lift of $u : M \to \mathbb{R}$ to the graph $\Gamma(u)$. Since u is proper and nonnegative, so is f. In addition, f is strictly convex by Lemma 3.4.1 since u is strictly convex. By Theorem 3.3.6, $\Gamma(u)$ is geodesically connected. ☐

3.5 Dual cones in semi-Euclidean space

In order to generate proper convex functions on timelike convex hypersurfaces in Minkowski space, we need to extend the notion of dual cones in Euclidean space to semi-Euclidean and Minkowski space. In Section 3.6 we apply the theory to convex hypersurfaces.

Definition 3.5.1. Let K be a subset of $\mathbb{E}^{n+k}_k$. The *dual cone* K^* of K in $\mathbb{E}^{n+k}_k$ is defined by $K^* = \{\mathbf{x}^* \in \mathbb{E}^{n+k}_k : \langle \mathbf{x}^*, \mathbf{x}\rangle \geq 0 \text{ for all } \mathbf{x} \in K\}$.

Proposition 3.5.2. *Let $K, K_1 \subseteq \mathbb{E}^{n+k}_k$.*

1. *K^* is a closed convex cone.*
2. *If $K_1 \subseteq K$, then $K_1^* \supseteq K^*$.*
3. *$(-K)^* = -K^*$.*
4. *If K has nonempty interior relative to $\mathbb{E}^{n+k}_k$, then K^* is pointed, i.e. K^* contains no line.*
5. *K^{**} is the closure of the smallest convex cone containing K.*
6. *Let ∂K denote the boundary of K relative to $\mathbb{E}^{n+k}_k$. If K is a convex cone, then $\mathbf{x} \in \partial K$ if and only if $\langle \mathbf{x}, \mathbf{x}^*\rangle = 0$ for some $\mathbf{x}^* \in K^*$.*

Proof. (1)–(5). Given a subset K of a finite dimensional vector space V, one can define the dual cone in the dual space as the linear functionals ℓ on V with $\ell(\mathbf{x}) \geq 0$ for $\mathbf{x} \in K$. These properties are well-known properties of this dual cone.

Equipping V with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ identifies V with its dual space. All linear functionals can be represented as $\ell_{\mathbf{w}}(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x}\rangle$ for $\mathbf{w}, \mathbf{x} \in V$. In this representation, the dual cone of a subset $K \subseteq V$ is $K^* = \{\mathbf{x}^* \in V : \langle \mathbf{x}^*, \mathbf{x}\rangle \geq 0\}$. Thus the same properties are carried over to the dual cone defined using the inner product, in particular if we take the semi-Euclidean inner product on $\mathbb{E}^{n+k}_k$.

(6). Suppose $\mathbf{x}_0 \in K - \partial K$. Then $\langle \mathbf{x}_0 + \mathbf{u}, \mathbf{x}^*\rangle \geq 0$ for all $\mathbf{u} \in U$ for some neighborhood U of 0 in $\mathbb{E}^{n+k}_k$ and for any $\mathbf{x}^* \in K^*$. For any $\mathbf{x}^* \in K^*$, choose $\mathbf{u} \in U$, $\langle \mathbf{u}, \mathbf{x}^*\rangle < 0$. Then $\langle \mathbf{x}_0, \mathbf{x}^*\rangle > 0$ for any $\mathbf{x}^* \in K^*$.

On the other hand, suppose $\mathbf{x}_0 \in \partial K$. Take $\mathbf{x}^* \in \mathbb{E}^{n+k}_k$ to be a nonzero normal vector to a supporting hyperplane of K at $\mathbf{x}_0 \in K$ with $\langle \mathbf{x} - \mathbf{x}_0, \mathbf{x}^*\rangle \geq 0$ for all $\mathbf{x} \in K$. Letting $\mathbf{x} = \lambda \mathbf{x}_0$ for any $\lambda > 0$, then $(\lambda - 1)\langle \mathbf{x}_0, \mathbf{x}^*\rangle \geq 0$. Letting $\lambda > 1$ and $\lambda < 1$ we obtain $\langle \mathbf{x}_0, \mathbf{x}^*\rangle = 0$. The claim follows. □

Proposition 3.5.3. *Let $\mathcal{F}$ and $\mathcal{P} = -\mathcal{F}$ denote the closed future and past cones in $\mathbb{E}^{n+1}_1$, respectively. Then $\mathcal{F}^* = \mathcal{P}$ and $\mathcal{P}^* = \mathcal{F}$.*

Proof. A simple calculation shows that $\mathbf{u} \in \mathcal{F}$ if and only if $\langle \mathbf{u}, \mathbf{w}\rangle \geq 0$ for all $\mathbf{w} \in \mathcal{P}$. □

Lemma 3.5.4. *Let $K \subseteq \mathbb{E}^{n+1}_1$ be a spacelike convex cone. Then either K is contained in a subspace of $\mathbb{E}^{n+1}_1$ of dimension $\leq n$ or K has nonempty interior relative to $\mathbb{E}^{n+1}_1$ and K^* contains a pair of linearly independent null vectors, $\mathbf{u} \in \mathcal{F}$ and $\mathbf{u}' \in \mathcal{P}$.*

Proof. Since K and $\mathcal{F}$ are convex and and intersect only at the origin, we can find a separating hyperplane $H_{\mathbf{w}} = \{\mathbf{x} \in \mathbb{E}_1^{n+1} : \langle \mathbf{x}, \mathbf{w} \rangle = 0\}$, $\mathbf{w} \neq \mathbf{0}$, such that $\mathbf{w} \in K^*$ and $-\mathbf{w} \in \mathcal{F}^* = \mathcal{P}$, i.e. $\mathbf{w} \in \mathcal{F}$. Thus $\mathbf{w} \in \mathcal{F} \cap K^*$. Similarly, we can find a nonzero vector $\mathbf{w}' \in \mathcal{P} \cap K^*$.

If $\mathbf{w}$ and $\mathbf{w}'$ are scalar multiples of one another, then K^* contains a line, and K lies in a subspace of dimension $\leq n$. Otherwise $\mathbf{w}$ and $\mathbf{w}'$ are linearly independent. Then the line segment between $\mathbf{w}$ and $\mathbf{w}'$ passes through a pair of linearly independent null vectors $\mathbf{u}$ and $\mathbf{u}'$, future-oriented and past-oriented respectively. Since K^* is convex, $\mathbf{u}, \mathbf{u}' \in K^*$. □

3.6 Convex hypersurfaces and geodesic connectedness

In this section, we prove Theorem 3.2.6 on geodesic connectedness of a timelike convex hypersurface M. The method is by constructing a convex function on M.

First we show that M is essentially the graph of a convex function over at least one of its tangent hyperplanes (Theorem 3.6.10). Wu proved the analogous theorem for Euclidean convex hypersurfaces in [W74]. In the Minkowski setting, the argument is somewhat more delicate (see Lemma 3.6.9 and Example 3.6.1).

The proof depends on Lemma 3.6.9 concerning the normal and recession cones of M. We begin with a few lemmas on normal and recession cones of general convex hypersurfaces in semi-Euclidean space. By a *general convex hypersurface* we will mean the boundary of a convex body, not necessarily smooth and not necessarily connected. (The latter provision merely allows the possibility of two parallel hyperplanes).

Unless otherwise specified, "interior" and the symbol "int" will mean interior relative to the original ambient semi-Euclidean space.

Definition 3.6.1. Let M be a general convex hypersurface of $\mathbb{E}_k^{n+k}$ bounding the convex body B in $\mathbb{E}_k^{n+k}$.

1. The *recession cone* $\mathcal{R}$ of M consists of all vectors on any ray from $\mathbf{0}$ in $\mathbb{E}_k^{n+k}$ that is the translate of a ray in B.
2. The *normal cone* $\mathcal{N}$ of M consists of all nonzero vectors $\mathbf{w} \in \mathbb{E}_k^{n+k}$ such that the half-space $\{\mathbf{x} \in \mathbb{E}_k^{n+k} : \langle \mathbf{x}, \mathbf{w} \rangle \geq 0\}$ is a translate of a supporting half-space of B at some $p \in M$, i.e. a half-space that contains B and whose boundary is tangent to B at p.

Definition 3.6.2. Given a choice of orthonormal basis of $\mathbb{E}_k^{n+k}$, the *associated Euclidean space* $\mathbb{E}^{n+k}$ is obtained by making the basis Euclidean orthonormal.

Remark 3.6.3. $\mathbf{w} = (w^1, ..., w^{n+k})$ is orthogonal to $\mathbf{w}_0$ in $\mathbb{E}_k^{n+k}$ if and only if $\mathbf{w}' = (w^1, ..., w^n, -w^{n+1}, ..., -w^{n+k})$ is orthogonal to $\mathbf{w}_0$ in the associated Euclidean space.

Lemma 3.6.4. *Let M be a general convex hypersurface in $\mathbb{E}_k^{n+k}$, and $\mathcal{N}$ be the normal cone of M. Then there exist a unique subspace $V \subseteq \mathbb{E}_k^{n+k}$ and a unique open convex cone K in V such that $K \subseteq \mathcal{N} \subseteq \overline{K}$, i.e. the closure and the interior relative to V of $\mathcal{N}$ are convex.*

Proof. Regard M as a convex hypersurface in an associated Euclidean space $\mathbb{E}^{n+k}$, and let $N : M \to S^{n+k-1}$ denote the Gauss map in $\mathbb{E}^{n+k}$. By Theorem 1 in [W74], there exist a unique totally geodesic sphere $S^m \subseteq S^{n+k-1}$ and a unique open convex subset U of S^m such that $U \subseteq N(M) \subseteq \overline{U}$.

For a set W in a vector space V, we set $\operatorname{ray} W = \{\lambda \mathbf{w} : \mathbf{w} \in W, \lambda \in [0, \infty)\}$. In $\mathbb{E}^{n+k}$, there is a one-to-one correspondence between open (closed) convex subsets of the unit sphere and open (closed) convex cones, obtained by identifying a point on the sphere with the open (closed) ray from the origin through that point. Thus there exist a unique subspace $V = \operatorname{ray} S^m$ in $\mathbb{E}^{n+k}$ and a unique open convex cone $K' = \operatorname{ray} U - \{\mathbf{0}\} \subseteq V$ such that $K' \subseteq \operatorname{ray} N(M) \subseteq \overline{K'}$.

Since $\mathbf{w} = (w^1, ..., w^{n+k})$ is an inward normal vector to M at a point p in $\mathbb{E}_k^{n+k}$ if and only if $\mathbf{w}' = (w^1, ..., w^n, -w^{n+1}, ..., -w^{n+k})$ is an inward normal vector to M at p in the associated Euclidean space $\mathbb{E}^{n+k}$, we have a vector space isomorphism mapping the normal cone $\mathcal{N}_{\mathrm{Euc}} = \operatorname{ray} N(M) - \{\mathbf{0}\}$ in the associated Euclidean space to the normal cone $\mathcal{N}$ in $\mathbb{E}_k^{n+k}$. All convex sets are carried to convex sets and the theorem follows. $\square$

Remark 3.6.5. By Lemma 3.6.4, the normal cone of a general convex hypersurface has convex interior relative to the subspace V, and convex closure. In [W74], an example of a smooth convex hypersurface M in $\mathbb{E}^3$ is described to show that the normal cone itself need not be convex.

To construct an analogous example in $\mathbb{E}_k^{n+k}$, consider an associated Euclidean space $\mathbb{E}^{n+k}$ and a convex hypersurface M in $\mathbb{E}^{n+k}$ whose normal cone $\mathcal{N}_{\mathrm{Euc}}$ is not convex. Since there is a vector space isomorphism $\mathbb{E}^{n+k} \to \mathbb{E}_k^{n+k}$ mapping the normal cone in the associated Euclidean space to the normal cone in $\mathbb{E}_k^{n+k}$, the normal cone $\mathcal{N}$ in $\mathbb{E}_k^{n+k}$ will not be convex.

Lemma 3.6.6. *Let M be a general convex hypersurface of $\mathbb{E}_k^{n+k}$ with recession cone $\mathcal{R}$ and normal cone $\mathcal{N}$. Then $\mathcal{N}^* = \mathcal{R}$ and $\mathcal{R}^* = \overline{\mathcal{N}}$.*

Proof. Let B be the convex body bounded by M. Suppose $\mathbf{u} \in \mathcal{R}$. Let $\mathbf{w} \in \mathcal{N}$ be a nonzero normal vector at $\mathbf{a} \in M$. Then $\mathbf{x} = \mathbf{u} + \mathbf{a} \in B$. By definition of $\mathcal{N}$, the hyperplane orthogonal to $\mathbf{w}$ supports B at $\mathbf{a}$, i.e. $\langle \mathbf{y} - \mathbf{a}, \mathbf{w} \rangle \geq 0$ for all $\mathbf{y} \in B$. In particular, $\langle \mathbf{x} - \mathbf{a}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle \geq 0$.

Suppose $\mathbf{u} \notin \mathcal{R}$. Choose $\mathbf{x} \in \operatorname{int} B$. The ray $\{\mathbf{x} + \lambda \mathbf{u} : \lambda > 0\}$ leaves B, say at $\mathbf{a} = \mathbf{x} + \lambda \mathbf{u} \in M$ for some $\lambda > 0$. Let $\mathbf{w} \in \mathcal{N}$ be a nonzero normal to M at $\mathbf{a}$. Then $\langle \mathbf{x} - \mathbf{a}, \mathbf{w} \rangle > 0$ since $\mathbf{x} \in \operatorname{int} B$, so $\langle -\lambda \mathbf{u}, \mathbf{w} \rangle > 0$ and $\langle \mathbf{u}, \mathbf{w} \rangle < 0$. Thus by Definition 3.5.1, $\mathcal{N}^* = \mathcal{R}$.

By Proposition 3.5.2, $\mathcal{R}^* = \mathcal{N}^{**}$ is the closure of the smallest convex cone containing $\mathcal{N}$. By Lemma 3.6.4, $\overline{\mathcal{N}}$ is the closure of the smallest convex cone containing $\mathcal{N}$, so $\mathcal{R}^* = \overline{\mathcal{N}}$. $\square$

Now we return to smooth convex hypersurfaces (Definition 3.2.5). We will use a definition of strong strict convexity that makes sense even at degenerate points, where second fundamental form is undefined:

Definition 3.6.7. Let M be a convex hypersurface of $\mathbb{E}^{n+k}_k$. We say $p \in M$ is a *point of weak strict convexity* if

$$M \cap T_pM = \{p\}.$$

We say $p \in M$ is a *point of strong strict convexity* if a neighborhood of p in M is the level set of a regular function that is defined on a neighborhood of p in $\mathbb{E}^{n+k}_k$ and has definite Hessian on T_pM. (Equivalently, p is a point of strong strict convexity in an associated Euclidean space.)

In light of the following lemma, we may speak of convex hypersurfaces *with a point of strict convexity* without specifying the type:

Lemma 3.6.8. *Let M be a convex hypersurface of $\mathbb{E}^{n+k}_k$. Then the following are equivalent:*

1. *M has a point of strong strict convexity,*
2. *M contains no line of $\mathbb{E}^{n+k}_k$,*
3. *M has a point of weak strict convexity.*

Proof. (2) $\Rightarrow$ (1): By Lemma 2 of [HN59] or Lemma 2 of [CL58], applied to the embedding of M in an associated Euclidean space $\mathbb{E}^{n+k}$, if there is no point of strong strict convexity then M contains a line.

(3) $\Rightarrow$ (2): If M contains a line, then M is ruled by parallel lines. Therefore M contains no point of weak strict convexity.

(1) $\Rightarrow$ (3): Obvious. $\square$

Lemma 3.6.9. *Suppose M is a timelike convex hypersurface in $\mathbb{E}^{n+1}_1$, $n \geq 2$, bounding the convex body B, and having a point of strict convexity. Let $\mathcal{R}$ and $\mathcal{N}$ denote the recession cone and normal cone of M, respectively. Then there is a nonzero vector $\mathbf{v}_0 \in (\operatorname{int} \mathcal{N}) \cap \mathcal{R}$.*

Proof. Since M has a point of strong strict convexity by Lemma 3.6.8, then $\operatorname{int} \mathcal{N} \neq \emptyset$.

Suppose $(\operatorname{int} \mathcal{N}) \cap \mathcal{R} = \emptyset$. Then the convex cones $\operatorname{int} \mathcal{N}$ and $\mathcal{R}$ are separated, i.e. lie in opposite closed half-spaces bounded by some n-dimensional subspace H. Let $\mathbf{w}$ be a nonzero normal vector to H, chosen so that $\langle \mathbf{w}, \mathbf{n} \rangle \leq 0$ for all $\mathbf{n} \in \mathcal{N}$ and $\langle \mathbf{w}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u} \in \mathcal{R}$. Then $\mathbf{w} \in -\mathcal{R}$ and $\mathbf{w} \in \mathcal{R}^* = \overline{\mathcal{N}}$ by

Lemma 3.6.6. Since $\mathcal{N}$ consists of spacelike vectors (because M is timelike), $\langle \mathbf{w}, \mathbf{w} \rangle \geq 0$, but since $\mathbf{w} \in -\mathcal{R}$, $\langle \mathbf{w}, \mathbf{w} \rangle \leq 0$. We conclude that $\mathbf{w}$ is null, so $\mathbf{w} \in \mathcal{F} \cup \mathcal{P}$.

Since $\operatorname{int} \mathcal{N}$ is a spacelike convex cone with nonempty interior relative to $\mathbb{E}_1^{n+1}$, we can apply Lemma 3.5.4 and choose a pair of linearly independent null vectors $\mathbf{u} \in \mathcal{F} \cap \mathcal{R}$ and $\mathbf{u}' \in \mathcal{P} \cap \mathcal{R}$. Since $\mathcal{R} = \mathcal{N}^* = \overline{\mathcal{N}}^*$, $\langle \mathbf{w}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{w}, \mathbf{u}' \rangle \geq 0$. However, this means that $\mathbf{w} \in \mathcal{F} \cap \mathcal{P} = \{\mathbf{0}\}$, a contradiction. $\square$

The following example shows that, in contrast to $\mathbb{E}^{n+1}$ [W74], for a non-timelike convex hypersurface in $\mathbb{E}_1^{n+1}$ with $\operatorname{int} \mathcal{N} \neq \emptyset$, $(\operatorname{int} \mathcal{N}) \cap \mathcal{R}$ can be empty.

Example 3.6.1. Let $B = \{(x, t) \in \mathbb{E}_1^2 : xt \geq 1, x > 0\}$ and $M = \partial B$. M is a strictly convex hypersurface in $\mathbb{E}_1^2$. The interior of the normal cone $\operatorname{int} \mathcal{N}$ is the open fourth quadrant and the recession cone $\mathcal{R}$ is the closed first quadrant, so $(\operatorname{int} \mathcal{N}) \cap \mathcal{R} = \emptyset$.

Theorem 3.6.10. *Suppose M is a timelike convex hypersurface in $\mathbb{E}_1^{n+1}$, $n \geq 2$, with a point of strict convexity. Then coordinates of $\mathbb{E}_1^{n+1}$ can be chosen so that the tangent hyperplane $T_{\mathbf{0}}M$ to M at the origin is $\{x^1 = 0\}$, and the following properties hold, where $\operatorname{int} D$ and ∂D denote the interior and boundary of D relative to $T_{\mathbf{0}}M$:*

1. *Let $\Pi : \mathbb{E}_1^{n+1} \to T_{\mathbf{0}}M$ be orthogonal projection, and D be the convex set $\Pi(M)$. Then over $\operatorname{int} D$, M is the graph of a convex function $u : \operatorname{int} D \to \mathbb{R}$.*

2. *For every $p \in D - \operatorname{int} D$, $M \cap \Pi^{-1}(p)$ is a closed spacelike half-line of $\mathbb{E}_1^{n+1}$.*

3. *For any $a > 0$, $L_a = M \cap \{x^1 = a\}$ is homeomorphic to S^{n-1}*

Proof. (1), (2), (3). Choose $\mathbf{v}_0 \in (\operatorname{int} \mathcal{N}) \cap \mathcal{R}$ as in Lemma 3.6.9, and linear coordinates on $\mathbb{E}_1^{n+1}$ so that the tangent hyperplane $T_{\mathbf{0}}M$ is $\{x^1 = 0\}$ and $\mathbf{v}_0 = (1, 0, ..., 0) \in \mathcal{N}$ is an inward normal to M at the origin. Since a compact convex hypersurface of $\mathbb{E}_1^{n+1}$ cannot have all tangent planes timelike, M is non-compact. If we regard M as embedded in the associated Euclidean space defined by these coordinates, M becomes the boundary of a convex body in $\mathbb{E}^{n+1}$ that contains no lines of $\mathbb{E}^{n+1}$.

By choice of $\mathbf{v}_0$, it remains true that $\mathbf{v}_0 \in (\operatorname{int} \mathcal{N}) \cap \mathcal{R}$ when $\mathcal{N}$ and $\mathcal{R}$ are defined in this associated $\mathbb{E}^{n+1}$. By Theorem 2 in [W74], (1), (2), and (3) hold because orthogonal projection in the associated $\mathbb{E}^{n+1}$ to $T_{\mathbf{0}}M$ is the same map as orthogonal projection in $\mathbb{E}_1^{n+1}$ to $T_{\mathbf{0}}M$. $\square$

Remark 3.6.11. Although M has a point of (strong or weak) strict convexity, it is not always possible to choose the coordinates so that the origin in Theorem 3.6.10 is such a point. Thus coordinates cannot always be chosen so that the critical set of u is a point. Locating an origin depends on Lemma 3.6.9 concerning $(\operatorname{int}\mathcal{N}) \cap \mathcal{R}$.

Theorem 3.6.12. *Suppose M is a timelike convex hypersurface of $\mathbb{E}_1^{n+1}$ with a point of strict convexity. Then M supports a proper nonnegative convex function f. If M is strongly strictly convex, then M supports a proper strictly convex function f.*

Proof. Consider coordinates on $\mathbb{E}_1^{n+1}$, projection $\Pi : \mathbb{E}_1^{n+1} \to T_{\mathbf{0}}M$, and $D = \Pi(M)$ as in Theorem 3.6.10. Set $f = x^1|_M = \langle \cdot, \mathbf{v}_0 \rangle$.

Let $\gamma : (a,b) \to M$ be a geodesic of M, and $N : M \to \mathbb{E}_1^{n+1}$ be the unit normal field on M with $N(p) \in \mathcal{N}$ for all $p \in M$.

The acceleration of γ in $\mathbb{E}_1^{n+1}$ can be written as $\gamma''(t) = \langle \gamma''(t), N_{\gamma(t)} \rangle N_{\gamma(t)}$, so $(f \circ \gamma)''(t) = \langle \gamma''(t), \mathbf{v}_0 \rangle = \langle \gamma''(t), N_{\gamma(t)} \rangle \langle N_{\gamma(t)}, \mathbf{v}_0 \rangle$. Since M is convex, the acceleration must be an inward normal vector at each point along γ, in the sense that $\langle \gamma''(t), N_{\gamma(t)} \rangle \geq 0$. Additionally, since $\mathbf{v}_0 \in \mathcal{R}$ and $N_{\gamma(t)} \in \mathcal{N}$, $\langle N_{\gamma(t)}, \mathbf{v}_0 \rangle \geq 0$. Thus, f is convex.

If M is strongly strictly convex, then $\gamma''(t) \neq 0$ and $\langle \gamma''(t), N_{\gamma(t)} \rangle > 0$ along γ. Moreover, the image of the Gauss map in the associated Euclidean space, and consequently its normal cone, is open and contains none of its boundary points. If $\langle N_{\gamma(t)}, \mathbf{v}_0 \rangle = 0$ for some t, then by (6) of Proposition 3.5.2, $\mathbf{v}_0$ is in the boundary of the normal cone, a contradiction, so $\langle N_{\gamma(t)}, \mathbf{v}_0 \rangle > 0$ along γ. We conclude that if M is strongly strictly convex, then $(f \circ \gamma)''(t) > 0$ along any non-constant geodesic, i.e. f is strictly convex.

Finally we show f is proper. Otherwise, there is some sublevel $M_a = \{p \in M : f(p) \leq a\}$ that is not compact. Then $\Pi(M_a)$ is a non-compact closed convex subset of D and has non-compact boundary $\partial\Pi(M_a) = \Pi(\partial M_a) = \Pi(L_a)$, contradicting compactness of L_a. □

Now we are ready to prove Theorem 3.2.6.

Theorem 3.6.13 (Theorem 3.2.6)**.** *Let M be a timelike convex hypersurface of $\mathbb{E}_1^{n+1}$. By splitting off a semi-Euclidean factor of maximal dimension, we may assume M is not ruled by parallel timelike or spacelike lines. Then M is geodesically connected if the following two conditions hold:*

1. *M does not contain an isometrically immersed Euclidean half-plane $\{(x,y) : x \in \mathbb{R}, y \geq 0\}$ where the half-lines $x =$ constant are carried to parallel half-lines of $\mathbb{E}_1^{n+1}$,*
2. *M is not ruled by parallel null lines of $\mathbb{E}_1^{n+1}$.*

Proof. Let k be the maximal dimension of a non-degenerate k-plane P contained in M. Then M contains through every point a translate of P. Identifying P with a coordinate subspace of $\mathbb{E}_1^{n+1}$, we have $M = M_0 \times P$, where M is embedded in $\mathbb{E}_1^{n+1} = P^\perp \times P$ as the product of a hypersurface embedding of M_0 in $P^\perp$ and the identity map of P. Thus M is geodesically connected if and only if M_0 is geodesically connected. If $P^\perp$ is Euclidean, then M_0 is Riemannian and complete, hence geodesically connected. Thus we need only consider timelike convex hypersurfaces M of $\mathbb{E}_1^{n+1}$ that are not ruled by parallel timelike or spacelike lines.

By hypothesis, M also is not ruled by parallel null lines. By Lemma 3.6.8, there is a point $p \in M$ of strict convexity.

Thus we may take M as described in Theorem 3.6.10. Consider the proper convex function $f : M \to \mathbb{R}$ given by $f = x^1|_M$, as in Theorem 3.6.12.

Claim 1. Any two points of the critical set C of f are joined by a geodesic of M, and C has an oriented neighborhood in M.

The critical points of $f = x^1|M$ are the points at which the n-plane tangent to M has the form $x^1 = c$. Since M bounds a convex body B, it follows that $C = M \cap \{x^1 = 0\} = B \cap \{x^1 = 0\}$, and C is a compact convex set.

A sufficiently small neighborhood of C in M is diffeomorphic by projection Π to a neighborhood of C in $T_{\mathbf{0}}M$, and hence is oriented.

Claim 2. For some non-critical value a, there is $p \in M_a - L_a$ such that the map $\psi_{(p,a)} : S_pM \to L_a$ has nonzero degree, where $L_a = M \cap \{x^1 = a\}$, $M_a = M \cap \{x^1 \leq a\}$, SM is the unit tangent bundle of M with respect to some choice of Riemannian metric, and $\psi_{(p,a)}(\mathbf{v})$ is the first point at which the geodesic of M with initial velocity $\mathbf{v} \in S_pM$ leaves M_a.

Choose $p \in C$. Since $\Pi(M_a)$ is convex in $\{x^1 = 0\}$, each geodesic in $\{x^1 = 0\}$ from p strikes $\Pi(L_a)$ transversely, and the corresponding map from S_pM to $\Pi(L_a)$ has degree 1. In $\{x^1 = 0\}$ and in M respectively, the vectors in SM tangent to geodesics from p agree on C. Since C is compact and M_a is arbitrarily close to C for a sufficiently small, it follows that each geodesic in M from p strikes L_a transversely when a is sufficiently small, and $\psi_{(p,a)}$ has degree 1.

Claim 3. M is disprisoning.

Since M is a topologically embedded timelike submanifold of $\mathbb{E}_1^{n+1}$ and $\mathbb{E}_1^{n+1}$ is strongly causal, M is strongly causal, as follows for instance from Lemma 3.4.2. Therefore the claim holds for non-spacelike geodesics [BEE96, Proposition 3.13].

Then M satisfies the hypotheses of Lemma 3.3.2. Accordingly if the claim fails there is a complete non-constant geodesic $\gamma : \mathbb{R} \to M$ such that $f \circ \gamma$ is constant. γ does not lie in the critical set C, since C is compact and the geodesics of C run on straight lines in $T_{\mathbf{0}}M$. Therefore γ lies in a level set L_a. Since L_a is compact and M is disprisoning on non-spacelike geodesics, it follows that γ is spacelike.

Moreover, γ cannot be a complete straight line. Let J be the nonempty open subset of $\mathbb{R}$ consisting of all t for which $\gamma''(t) \neq \mathbf{0}$. For $t \in J$,

$$\gamma''(t) \in \{x^1 = a\} \cap (T_{\gamma(t)}M)^{\perp}.$$

Thus $T_{\gamma(t)}M$ is vertical, i.e. in the notation of Theorem 3.6.10 (1),

$$(\Pi \circ \gamma)|_J \subset D - \operatorname{int} D.$$

By Theorem 3.6.10 (2), $M \cap \Pi^{-1}(\gamma(t))$ contains the closed vertical half-line with endpoint $\gamma(t)$ and lying above $\gamma(t)$, for all t in the closure of J, $\overline{J}$.

Let I be any maximal nonempty open subinterval of $\mathbb{R} - \overline{J}$. By compactness of L_a, I is a finite interval. If t is an endpoint of I, then $M \cap T_{\gamma(t)}M$ contains the line segment $\gamma(I)$ and the vertical half-line above $\gamma(t)$. Thus since B is convex, the vertical planar strip above $\gamma(I)$ lies in $B \cap T_{\gamma(t)}M$, and in fact lies in $M \cap T_{\gamma(t)}M$ since $T_{\gamma(t)}M$ is a support hyperplane. It follows that assumption (1) is contradicted, proving claim 3.

By Lemma 3.3.5, the theorem follows from these three claims. □

Remark 3.6.14. We do not know if condition (1) of Theorem 3.2.6 is necessary. A recent paper of Bartolo, Candela, and Flores, giving a sufficient condition for a globally hyperbolic space-time with a null Killing vector field to be geodesically connected [BCF14], can be used to eliminate condition (2). See Section 3.9.

Now let us verify a claim from Section 3.1:

Proposition 3.6.15. *The timelike convex hypersurfaces M of $\mathbb{E}^{n+1}_1$ satisfy $\mathcal{R} \geq 0$.*

Proof. If

$$\mathrm{II} : T_pM \times T_pM \to (T_pM)^{\perp}$$

is the second fundamental form of M in $\mathbb{E}^{n+1}_1$, then the Gauss Equation states

$$\langle R(\mathbf{x}, \mathbf{y})\mathbf{x}, \mathbf{y}\rangle = \langle \mathrm{II}(\mathbf{x}, \mathbf{x}), \mathrm{II}(\mathbf{y}, \mathbf{y})\rangle - \langle \mathrm{II}(\mathbf{x}, \mathbf{y}), \mathrm{II}(\mathbf{x}, \mathbf{y})\rangle. \tag{3.3}$$

Locally, a timelike convex hypersurface is the graph of a convex Lorentzian function $f : U \to \mathbb{R}$ where U is a neighborhood of $\mathbf{0} \in T_pM \cong \mathbb{E}^n_1$. A simple calculation yields

$$\langle R(\mathbf{x}, \mathbf{y})\mathbf{x}, \mathbf{y}\rangle = \frac{\nabla^2 f(\mathbf{x}, \mathbf{x})\nabla^2 f(\mathbf{y}, \mathbf{y}) - \nabla^2 f(\mathbf{x}, \mathbf{y})^2}{1 + \langle (\nabla f)_p, (\nabla f)_p\rangle} \tag{3.4}$$

The numerator is nonnegative by convexity of f and the denominator is positive because f is Lorentzian. It follows that timelike sectional curvatures are ≤ 0 and spacelike sectional curvatures are ≥ 0. □

3.7 Existence criterion for convex functions

The aim of this section is to prove a criterion for the non-critical level sets of a (proper) not-necessarily-convex function $u : M \to \mathbb{R}$ on a semi-Riemannian manifold to be the levels of a (proper) convex function $f : M \to \mathbb{R}$. We use this criterion to extend our theorems on geodesic connectedness. Finally we apply these results to a class of not-necessarily-convex hypersurfaces in Minkowski space.

The corresponding Riemannian criterion is given in [AB74].

Theorem 3.7.1. *Suppose $u : M \to \mathbb{R}$ is a smooth function on a semi-Riemannian manifold M. Denote the critical set of u by C. For every $a \in \operatorname{range} u$, we denote the non-critical points of $u^{-1}(a)$ by L_a.*

Suppose N is a vector field on $M - C$ satisfying $Nu > 0$. Define $\eta : M - C \to \mathbb{R}$ by

$$\eta(p) = \nabla^2 u(N_p, N_p)/(N_p u)^2.$$

Define $\mu : \{a \in \mathbb{R} : L_a \neq \emptyset\} \to \mathbb{R}$ by

$$\mu(a) = \inf \big\{\eta(p), \eta(p) - \big(\nabla^2 u(\mathbf{x}, N_p)^2/(N_p u)^2 \nabla^2 u(\mathbf{x}, \mathbf{x})\big) : \\ p \in L_a, \mathbf{x} \in T_p L_a, \nabla^2 u(\mathbf{x}, \mathbf{x}) \neq 0\big\}. \tag{3.5}$$

Then there is a smooth function $\rho : \operatorname{range} u \to \mathbb{R}$ such that $\rho' \geq 1$ and $f = \rho \circ u$ is convex if and only if u satisfies the following conditions (1) - (4):

1. *the critical set C consists of local minimum points,*
2. *the restriction of $\nabla^2 u$ to each tangent space $T_p L_a$ is positive semidefinite (i.e. each L_a is a locally convex hypersurface on which N is outward-pointing),*
3. *if $\mathbf{x} \in T_p L_a$ and $\nabla^2 u(\mathbf{x}, \mathbf{x}) = 0$, then $\nabla^2 u(\mathbf{x}, N_p) = 0$,*
4. *the function μ is bounded below by a function that extends to a continuous function $h : \operatorname{range} u \to \mathbb{R}$ (which clearly may be assumed non-positive).*

Moreover, $f = \rho \circ u$ is strictly convex if and only if u satisfies condition (4) and the following conditions (i) and (ii):

(i) the critical points are non-degenerate local minima,

(ii) the restriction of $\nabla^2 u$ to each tangent space $T_p L_a$ is positive definite.

Finally, if u is proper, then f may be assumed proper.

Remark 3.7.2. Since $\nabla^2 u$ is semidefinite at each $p \in L_a$, condition (3) means the nullspace of $\nabla^2 u$ on $T_p L_a$ lies in the nullspace of $\nabla^2 u$.

Remark 3.7.3. As mentioned in the introduction, it is often possible to verify (4) by showing that μ is continuous and finite.

Proof. In [AB74, Theorem 1], Alexander and Bishop proved this result when M is a Riemannian manifold and $N = \nabla u/\|\nabla u\|$, the outward unit normal field to the level sets of u. However, it suffices to require only $Nu > 0$ and M semi-Riemannian. Then the calculations in the proof are unchanged.

In particular, the convexity condition of f is shown to imply that we may take ρ to be a solution of the differential equation $\rho'' + h\rho' = 0$ on range u. Since h is non-positive, ρ can be chosen so that $\rho'(a) = \exp(-\int_{a_0}^{a} h) \geq 1$ where $a_0 = \inf \operatorname{range} u$. Therefore ρ is proper. Thus f will be proper if u is proper. $\square$

Now we have the following analog of Theorem 3.3.6 and Corollary 3.4.3:

Theorem 3.7.4. *Let M be a null-disprisoning semi-Riemannian manifold. Suppose M supports a proper nonnegative function $u : M \to \mathbb{R}$ whose critical set is a point and for which there is no non-constant complete geodesic on which u is constant.*

Suppose u satisfies conditions (1),(2),(3),(4) of Theorem 3.7.1 for a vector field N on $M - C$ satisfying $Nu > 0$. Then M is geodesically connected.

If in addition M is a strongly causal space-time and u is a Lorentzian function, then the graph $\Gamma(u)$ is geodesically connected.

Proof. Since $u : M \to \mathbb{R}$ satisfies the conditions of Theorem 3.7.1 for the vector field N, we can choose a smooth function $\rho : \operatorname{range} u \to \mathbb{R}$ such that the function $f_M = \rho \circ u$ is proper, convex, and has the same critical set, level sets, and sublevel sets as u. Since the critical set of f_M is a point, and there is no non-constant complete geodesic on which f_M is constant, it follows from Theorem 3.3.6 that M is geodesically connected.

Now assume further that M is a strongly causal space-time and u is a Lorentzian function. By Lemma 3.4.2, $\Gamma(u)$ is strongly causal. Thus $\Gamma(u)$ is null-disprisoning [BEE96, Proposition 3.13].

Since projection $\Pi : \Gamma(u) \to M$ by $(p, u(p)) \mapsto p$ is a diffeomorphism, we may identify a vector $\mathbf{x} \in T_pM$ with its lift, $\mathbf{x} + (\mathbf{x}u)\partial_y$, via the inverse projection map. Here we write $\mathbf{x}$ for either. We write $v : \Gamma(u) \to \mathbb{R}$ for the lift of u to $\Gamma(u)$. Thus $Nu = Nv > 0$. The non-critical level sets of u and their lifts to level sets of v will both be denoted by L_a, and similarly for the sublevel sets M_a and the critical set C. Thus C denotes either the minimum point of u on M or the minimum point of v on $\Gamma(u)$.

Claim 1. The lift $v : \Gamma(u) \to \mathbb{R}$ of u to $\Gamma(u)$ satisfies the conditions of Theorem 3.7.1. for the vector field N on $\Gamma(u) - C$. Thus there is a smooth function $\rho : \operatorname{range} v \to \mathbb{R}$ such that $\rho' \geq 1$ and $f = \rho \circ v : \Gamma(u) \to \mathbb{R}$ is proper and convex with the same level sets and critical sets as v.

Since u is Lorentzian, $1 + \langle \nabla u, \nabla u \rangle > 0$ and by Lemma 3.4.1,

$$\nabla^2 v(\mathbf{x}, \mathbf{y}) = \frac{\nabla^2 u(\mathbf{x}, \mathbf{y})}{1 + \langle (\nabla u)_p, (\nabla u)_p \rangle}.$$

Thus $\nabla^2 v(\mathbf{x},\mathbf{x}) \geq 0$ if and only if $\nabla^2 u(\mathbf{x},\mathbf{x}) \geq 0$. Since L_a is infinitesimally convex in M for any a, L_a is also infinitesimally convex in $\Gamma(u)$.

Furthermore, $\nabla^2 v(\mathbf{x},\mathbf{y}) = 0$ if and only if $\nabla^2 u(\mathbf{x},\mathbf{y}) = 0$. If $\nabla^2 v(\mathbf{x},\mathbf{x}) = 0$, then $\nabla^2 u(\mathbf{x},\mathbf{x}) = 0$ and $\nabla^2 u(\mathbf{x}, N_p) = 0$ by condition (3), and hence $\nabla^2 v(\mathbf{x}, N_p) = 0$.

Define $\eta^u : M - C \to \mathbb{R}$ and $\mu^u : \operatorname{range}(u|_{M-C}) \to \mathbb{R}$ as η and μ are defined for u in Theorem 3.7.1, and $\eta^v : \Gamma(u) - C \to \mathbb{R}$ and $\mu^v : \operatorname{range}(v|_{\Gamma(u)-C}) \to \mathbb{R}$ similarly for v. Let $h^u : \operatorname{range} u \to \mathbb{R}$ be a continuous lower bound of μ^u. Then by condition (4),

$$\eta^v(p) = \nabla^2 v(N_p, N_p)/(N_p v)^2 = \frac{\nabla^2 u(N_p, N_p)/(N_p u)^2}{1 + \langle (\nabla u)_p, (\nabla u)_p \rangle} \geq h^u(a)/w(a)$$

where $p \in M_a$ and $w(a) = \sup\{1 + \langle (\nabla u)_q, (\nabla u)_q \rangle : q \in M_a\}$. Similarly, if $\nabla^2 v(\mathbf{x},\mathbf{x}) \neq 0$, then

$$\eta^v(p) - \frac{\nabla^2 v(\mathbf{x}, N_p)^2}{(N_p v)^2 \nabla^2 v(\mathbf{x},\mathbf{x})} \geq h^u(a)/w(a).$$

Since u is Lorentzian, $w(a)$ is a positive nondecreasing continuous function. Thus $h^v(a) := h^u(a)/w(a)$ is a continuous lower bound of $\mu^v(a)$ on $\operatorname{range} v$, and claim 1 is verified.

Suppose γ is a non-constant complete geodesic of $\Gamma(u)$ on which f is constant. Since γ is a curve in the graph of u, we can find $\alpha : \mathbb{R} \to M$ so that $\gamma = (\alpha, u \circ \alpha)$ is the lift of α to $\Gamma(u)$ and as in (3.2),

$$\alpha''(t) + (u \circ \alpha)''(t)(\nabla u)_{\alpha(t)} = \mathbf{0}.$$

However, since γ is in a level set of f and therefore of v, α must be in a level set of u. Thus $(u \circ \alpha)'' = 0$, hence $\alpha'' = \mathbf{0}$. Thus α is a non-constant complete geodesic of M on which u is constant, a contradiction. Therefore by Theorem 3.3.6, $\Gamma(u)$ is geodesically connected. $\square$

Finally we construct a large class of not-necessarily-convex but geodesically connected Lorentzian hypersurfaces. Specifically, we may perturb the levels of a strictly convex Lorentzian function by any σ satisfying $0 < \sigma' \leq 1$ and still retain a geodesically connected graph:

Corollary 3.7.5. *Let $\sigma : [0,\infty) \to [0,\infty)$ be a proper smooth function with $0 < \sigma' \leq 1$ and let $f : \mathbb{E}^n_1 \to \mathbb{R}$ be a proper nonnegative strictly convex Lorentzian function. Let $M \subset \mathbb{E}^{n+1}_1$ be the graph of $u = \sigma \circ f : \mathbb{E}^n_1 \to \mathbb{R}$ in $\mathbb{E}^n_1 \times \mathbb{R} = \mathbb{E}^{n+1}_1$. Then M is a timelike hypersurface of $\mathbb{E}^{n+1}_1$ and is geodesically connected.*

Proof. u is Lorentzian since

$$\langle \nabla u, \nabla u \rangle = \langle \nabla f, \nabla f \rangle (\sigma' \circ f)^2 > (-1)(\sigma' \circ f)^2 \geq -1.$$

Let $\rho = \sigma^{-1}$ and $N = \nabla_{\mathrm{Riem}} u$, the gradient of u in any Riemannian metric g_{Riem} on M. Since $f = \rho \circ u$ is strictly convex, there is no non-constant complete geodesic on which u is constant. Since $\rho' = 1/(\sigma' \circ \sigma^{-1}) \geq 1$ and $Nu > 0$ on the non-critical set, then by Theorem 3.7.1, u satisfies (1), (2), (3) and (4) for N. By Theorem 3.7.4, since $\mathbb{E}^n_1$ is null-disprisoning, M is geodesically connected. □

3.8 Orthogonal splittings

Let us recall the definition of an orthogonal splitting space-time and state the main theorem on geodesic connectedness in the survey by Candela and Sanchez [CS08, Theorem 4.37]:

Definition 3.8.1. A Lorentzian manifold M is an *orthogonal splitting space-time* if M is isometric to $(M_0 \times \mathbb{R}, \langle \cdot, \cdot \rangle_L)$ where

$$\langle \mathbf{z}, \mathbf{z}' \rangle_L = \langle A(z)\mathbf{x}, \mathbf{x}' \rangle_R - \beta(z)\mathbf{t}\mathbf{t}' \tag{3.6}$$

for any $z = (x, \tau) \in M$ $(x \in M_0, \tau \in \mathbb{R})$, and $\mathbf{z} = (\mathbf{x}, \mathbf{t}), \mathbf{z}' = (\mathbf{x}', \mathbf{t}') \in T_z M \cong T_x M_0 \times \mathbb{R}$. Here $(M_0, \langle \cdot, \cdot \rangle_R)$ is a finite-dimensional, connected Riemannian manifold, $A(z) : T_x M_0 \to T_x M_0$ is a smooth, symmetric, strictly positive linear operator, and $\beta : M \to \mathbb{R}$ is a smooth, strictly positive scalar field.

Theorem 3.8.2. *Let M be an orthogonal splitting space-time, isometric to $(M_0 \times \mathbb{R}, \langle \cdot, \cdot \rangle_L)$, where $(M_0, \langle \cdot, \cdot \rangle_R)$ is a complete Riemannian manifold. Assume that there exist constants a, b, c, d > 0 such that the coefficients A, β in (3.8.1) satisfy the following hypotheses:*

$$a\langle \mathbf{x}, \mathbf{x} \rangle_R \leq \langle A(z)\mathbf{x}, \mathbf{x} \rangle_R, \tag{3.7}$$

$$b \leq \beta(z) \leq c, \tag{3.8}$$

$$|\beta_\tau(z)| \leq d, \quad |\langle A_\tau(z)\mathbf{x}, \mathbf{x} \rangle_R| \leq d\langle \mathbf{x}, \mathbf{x} \rangle_R, \tag{3.9}$$

for all $z = (x, \tau) \in M$, $\mathbf{x} \in T_x M_0$. Furthermore, assume that

$$\limsup_{\tau \to +\infty} (\sup\{\langle A_\tau(z)\mathbf{x}, \mathbf{x} \rangle_R : x \in M_0, \mathbf{x} \in T_x M_0, \langle \mathbf{x}, \mathbf{x} \rangle_R = 1\}) \leq 0, \tag{3.10}$$

$$\liminf_{\tau \to -\infty} (\inf\{\langle A_\tau(z)\mathbf{x}, \mathbf{x} \rangle_R : x \in M_0, \mathbf{x} \in T_x M_0, \langle \mathbf{x}, \mathbf{x} \rangle_R = 1\}) \geq 0. \tag{3.11}$$

Then M is geodesically connected.

In this section, we examine the conditions of Theorem 3.8.2 in two natural examples of splittings of a strictly convex hypersurface M. We take M to be the graph in $\mathbb{E}^3_1$ given by

$$x^2 = f(x^1, t) = \sqrt{(x^1)^2 + t^2 + 1}.$$

By Theorem 3.2.6, M is geodesically connected. We show that the splittings do not satisfy the conditions of Theorem 3.8.2, nor do we know of any other splittings that do so.

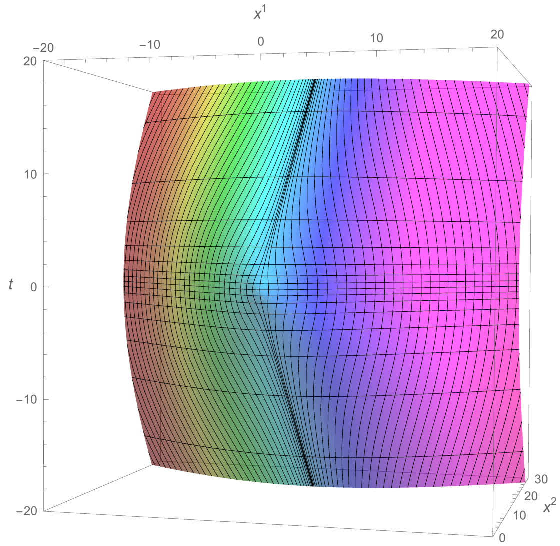

Figure 3.1: The timelike convex hypersurface M with orthogonal splitting coordinates generated by taking $\tau((x^1, f(x^1,t),t)) = \sinh^{-1}(t)$ as time function.

Figure 3.1 illustrates the first example. Intersecting M with hyperplanes $t =$ constant gives a family of Cauchy hypersurfaces which we parametrize with a time function τ as follows:

$$M_\tau = M \cap \{(x^1, x^2, t) \in \mathbb{E}^3_1 : t = \sinh(\tau)\}.$$

The orthogonal trajectories $\gamma_x(t)$ are illustrated, obtained by solving a family of ordinary differential equations numerically. In order for the coordinates

$$(x, \tau) \mapsto (x^1(x,\tau), f(x^1(x,\tau), \sinh(\tau)), \sinh(\tau))$$

to form an orthogonal splitting, x^1 must satisfiy the differential equation

$$x^1_\tau(x,\tau) = -\frac{x^1(x,\tau)\sinh(\tau)\cosh(\tau)}{\cosh^2(\tau) + 2(x^1(x,\tau))^2}.$$

Then

$$\beta(x,\tau) = \frac{(1 + 2(x^1(x,\tau))^2)\cosh^2(\tau)}{2(x^1(x,\tau))^2 + \cosh^2(\tau)}.$$

Since β is unbounded along the curve $x^1 = \cosh(\tau)$, hypothesis (3.8) of Theorem 3.8.2 is violated. Additionally, one can show that $A(x,\tau) \to 0$ as $\tau \to \pm\infty$ along the meridian $x^1 = 0$, so the condition $a > 0$ on A in hypothesis (3.7) is violated. The splitting can be modified by reparametrizing the time function τ but this does not effect the boundedness of A.

Another natural splitting, illustrated in Figure 3.2, is obtained by boosting

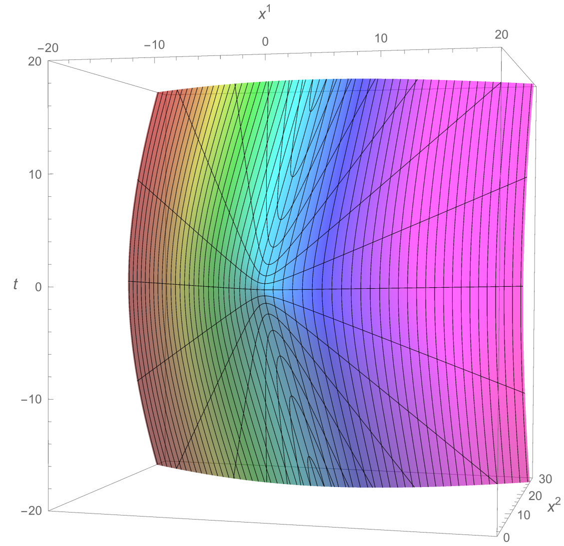

Figure 3.2: The timelike convex hypersurface M with orthogonal splitting coordinates given by boost coordinates.

the $t = 0$ slice about the x^1-axis, namely

$$(x, \tau) \mapsto (x, \cosh(\tau)\sqrt{1+x^2}, \sinh(\tau)\sqrt{1+x^2}).$$

We obtain $A(x, \tau)\mathbf{x} = ((1+2x^2)/(1+x^2))\mathbf{x}$ and $\beta(x, \tau) = 1 + x^2$. Thus β is unbounded above as $x \to \pm\infty$. This splitting can be modified by reparametrizing the time function τ but β will remain unbounded as $x \to \pm\infty$. With this particular splitting the metric is static, i.e. A and β are independent of τ, and other methods discussed in [CS08] can be applied to show geodesic connectedness.

However, there may well exist non-static hypersurfaces satisfying the conditions of our main theorem, on which an orthogonal splitting metric cannot simultaneously satisfy the required bounds on A and β. In general it is unclear how to know if a given timelike hypersurface is static or has orthogonal splittings satisfying the conditions of Theorem 3.8.2.

3.9 Null-ruled timelike convex hypersurfaces of Minkowski space

In Remark 3.6.14, we noted that a recent paper of Bartolo, Candela, and Flores gave sufficient conditions for a globally hyperbolic space-time with a null Killing vector field to be geodesically connected [BCF14]. This result can be used to establish geodesic connectedness for timelike convex hypersurfaces in Minkowski space ruled by parallel null lines.

In an initial investigation of the use of Killing fields to establish geodesic connectedness of Lorentzian manifolds, Candela, Flores, and Sánchez proved the following.

Theorem 3.9.1. *[CFS08, Theorem 1.1] Let $(L, \langle\cdot,\cdot\rangle_L)$ be a stationary space-time with a complete* timelike *Killing vector field K. If L is globally hyperbolic with a complete (smooth, spacelike) Cauchy hypersurface Σ, then it is geodesically connected.*

Their proof involves using the infinite dimensional variational methods discussed in [M94]. In [BCF14], Bartolo, Candela, and Flores extended these results to space-times supporting null Killing vector fields with the following result.

Theorem 3.9.2. *[BCF14, Theorem 1.2] Let $(L, \langle\cdot,\cdot\rangle_L)$ be a globally hyperbolic space-time endowed with a complete* null *Killing vector field K and a complete (smooth, spacelike) Cauchy hypersurface Σ. Given two points $p, q \in L$, the following statements are equivalent:*

(i) p and q are geodesically connected in L;

(ii) p and q can be connected by a C^1 curve φ on L such that $\langle\varphi', K\circ\varphi\rangle_L$ has constant sign or is identically equal to 0.

In Theorem 3.6.13, we proved that a large class of timelike convex hypersurfaces of $\mathbb{E}_1^{n+1}$ are geodesically connected, but condition 2 excludes timelike convex hypersurfaces M ruled by parallel null lines. When M is ruled by spacelike or timelike lines, we can split them off and they do not affect the geodesic connectedness of M. When M is ruled by parallel null lines, there is no similar splitting procedure. However, in this scenario we can construct a null Killing vector field on M (by taking a constant vector field K on Minkowski space whose restriction to M points in the direction of the parallel null lines), and apply Theorem 3.9.2.

Example 2.2.5 is a simple three-dimensional example of a timelike convex hypersurface of $\mathbb{E}_1^4$ ruled by parallel null lines. The example is obtained by taking the paraboloid $\{z = x^2 + y^2\}$ in the spacelike hyperplane $\{t = 0\}$ and translating it at the speed of light in the x-direction to obtain

$$L = \{(x, y, z, t) \in \mathbb{E}_1^4 : z = (x-t)^2 + y^2\}.$$

In the following proposition, we generalize this construction and prove that all timelike hypersurfaces of $\mathbb{E}_1^{n+1}$ ruled by parallel null lines can be constructed this way.

Proposition 3.9.3. *Let $\mathbf{v} \in S^{n-1}$ and $C \subseteq \mathbb{E}^n$ be a hypersurface of Euclidean space having no tangent hyperplane orthogonal to $\mathbf{v}$. Define the hypersurface $L_{C,\mathbf{v}}$ of $\mathbb{E}_1^{n+1}$ by*

$$L_{C,\mathbf{v}} = \{(p + t\mathbf{v}, t) \in \mathbb{E}^n \times (-\mathbb{R}) = \mathbb{E}_1^{n+1} : p \in C, t \in \mathbb{R}\}. \tag{3.12}$$

Then $L_{C,\mathbf{v}}$ is a timelike hypersurface of $\mathbb{E}_1^{n+1}$ ruled by null lines. If C is a convex hypersurface of $\mathbb{E}^n$, then $L_{C,\mathbf{v}}$ is a convex hypersurface of $\mathbb{E}_1^{n+1}$.

Furthermore, any timelike hypersurface M of $\mathbb{E}_1^{n+1}$ that is ruled by parallel null lines can be constructed in this way, for some $\mathbf{v} \in S^{n-1}$ and some hypersurface $C \subseteq \mathbb{E}^n$ having no tangent plane orthogonal to $\mathbf{v}$. M is a convex hypersurface if and only if C is a convex hypersurface.

Proof. We will denote $L_{C,\mathbf{v}}$ as L throughout the proof. Let N be a local normal vector field on C. A simple calculation shows that $N_L(p + t\mathbf{v}, t) = N(p) + \langle N(p), \mathbf{v}\rangle \frac{\partial}{\partial t}$ is a local normal vector field on L. L is timelike if the normal field to L is spacelike, i.e.

$$\langle N_L, N_L\rangle = \langle N, N\rangle - \langle N, \mathbf{v}\rangle^2 > 0.$$

We know from the Cauchy-Schwartz inequality that $\langle N, \mathbf{v}\rangle^2 \leq \langle N, N\rangle\langle \mathbf{v}, \mathbf{v}\rangle = \langle N, N\rangle$ with equality only if N and $\mathbf{v}$ are parallel. Thus, $\langle N_L, N_L\rangle > 0$ and L is timelike.

If C is the boundary of a convex body B in $\mathbb{E}^n \subseteq \mathbb{E}_1^{n+1}$, then L is the boundary of the convex body $\{(\mathbf{x}, 0) + s(\mathbf{v}, 1) : \mathbf{x} \in B, s \in \mathbb{R}\}$ in $\mathbb{E}_1^{n+1}$, where $(\mathbf{v}, 1)$ is the null vector whose orthogonal projection into $\mathbb{E}^n$ is $\mathbf{v}$. Additionally L is ruled by the parallel null lines $\{(p, 0) + s(\mathbf{v}, 1) : s \in \mathbb{R}\}$ for $p \in C$.

To see that any timelike hypersurface M of $\mathbb{E}_1^{n+1}$ ruled by parallel null lines can be constructed in this way, choose $\mathbf{v}$ so that $\mathbf{v} + \frac{\partial}{\partial t}$ is tangent to the parallel null lines associated to M and let $C = M \cap \mathbb{E}^n$ where $\mathbb{E}^n$ is the spacelike hyperplane $\{x^{n+1} = 0\}$. It is obvious that C is a convex hypersurface of $\mathbb{E}^n$ if and only if M is a convex hypersurface of $\mathbb{E}_1^{n+1}$. $\square$

For $\mathbf{v} \in S^{n-1}$ and a convex hypersurface $C \subseteq \mathbb{E}^n$ as above, let us denote the hypersurface L as defined in (3.12) by $L_{C,\mathbf{v}}$. We now apply Theorem 3.9.2 to $L_{C,\mathbf{v}}$ to obtain a condition equivalent to geodesic connectedness for $L_{C,\mathbf{v}}$.

Lemma 3.9.4. *Let $L_{C,\mathbf{v}}$ be the timelike hypersurface of $\mathbb{E}_1^{n+1}$ defined by (3.12), where $\mathbf{v} \in S^{n-1}$ and C is a hypersurface which is nowhere orthogonal to $\mathbf{v}$. Then the following statements are equivalent:*

(i) $L_{C,\mathbf{v}}$ is geodesically connected;

(ii) for any pair of points $p, q \in C$, p and q can be connected by a C^1 curve α on C such that $\langle \alpha', \mathbf{v}\rangle$ has constant sign or is identically equal to 0.

Proof. Note that $K = (\mathbf{v} + \frac{\partial}{\partial t})|_{L_{C,\mathbf{v}}}$ is a null Killing field on $L_{C,\mathbf{v}}$. K is complete since its integral curves are the null lines that rule $L_{C,\mathbf{v}}$.

Additionally, $L_{C,\mathbf{v}}$ inherits global hyperbolicity from $\mathbb{E}_1^{n+1}$. To see this, let Σ be the intersection of any Cauchy hypersurface of $\mathbb{E}_1^{n+1}$ with $L_{C,\mathbf{v}}$. Any inextendible causal curve γ of $L_{C,\mathbf{v}}$ will be causal and inextendible in $\mathbb{E}_1^{n+1}$

since $L_{C,\mathbf{v}}$ is closed in $\mathbb{E}^{n+1}_1$. Thus Σ is a Cauchy hypersurface $L_{C,\mathbf{v}}$ and $L_{C,\mathbf{v}}$ is globally hyperbolic.

By Theorem 3.9.2, a pair of points $p, q \in L_{C,\mathbf{v}}$ are connected by a geodesic if and only if there is a curve φ from p to q such that the sign of $\langle \varphi', K \circ \varphi\rangle$ is constant or it is identically equal to 0.

Given a curve φ in $L_{C,\mathbf{v}}$, we can write φ, using a pair of curves $\alpha : I \to C$ and $\beta : I \to \mathbb{R}$, as $\varphi(s) = (\alpha(s) + \mathbf{v}\beta(s), \beta(s))$. Thus,

$$\langle \varphi', K \circ \varphi\rangle = \langle (\alpha' + (\mathbf{v} + \tfrac{\partial}{\partial t})\beta', \mathbf{v} + \tfrac{\partial}{\partial t}\rangle = \langle \alpha', \mathbf{v}\rangle.$$

For a pair of points $(p + t_p\mathbf{v}, t_p), (q + t_q v, t_q) \in L_{C,\mathbf{v}}$, let β be any curve from t_p to t_q. If there is a curve α in C from p to q such that $\langle \alpha', \mathbf{v}\rangle$ has constant sign or vanishes, then there is a geodesic connecting $(p + t_p\mathbf{v}, t_p)$ to $(q + t_q\mathbf{v}, t_q)$ in $L_{C,\mathbf{v}}$. Hence the lemma. $\square$

If C is the graph of a convex function on a hyperplane of $\mathbb{E}^n$ containing $\mathbf{v}$ (e.g. as in Example 2.2.5), property (ii) in Lemma 3.9.4 can be easily shown to hold. The following theorem shows that for any such pair C and $\mathbf{v}$, property (ii) holds and thus all timelike convex hypersurfaces containing a null line are geodesically connected.

Theorem 3.9.5. *Any timelike convex hypersurface L of $\mathbb{E}^{n+1}_1$ containing a null line of $\mathbb{E}^{n+1}_1$ is geodesically connected.*

Proof. First, write $L = L_{C,\mathbf{v}}$ as the null-ruled surface generated by a convex hypersurface C in $\mathbb{E}^n$ and $\mathbf{v} \in S^{n-1}$. We proceed by showing that between any pair of points $p, q \in C$, there exists a C^1 curve $\alpha : [0, 1] \to C$ such that $\langle \alpha', \mathbf{v}\rangle$ is constant along α.

Let k be the maximal dimension of a k-plane P contained in C. Then C contains through every point a translate of P. Identifying P with a coordinate subspace of $\mathbb{E}^n$, we have $C = C_0 \times P$, where C is embedded in $\mathbb{E}^n = P^\perp \times P$ as the product of a hypersurface embedding of C_0 in $P^\perp$ and the identity map of P. For a pair of points $p, q \in C_0$, if we can construct a curve α from p to q in C_0 with the desired property, then C trivially inherits the same property. Thus, without loss of generality, we may assume that C contains no lines.

Let B be the convex body in $\mathbb{E}^n$ whose boundary is C and fix a pair of points $p, q \in C$. Denote by H_p and H_q the hyperplanes of $\mathbb{E}^n$ orthogonal to $\mathbf{v}$ containing p and q, respectively. $C \cap H_p$ and $C \cap H_q$ are convex hypersurfaces of H_p and H_q, respectively. Let $\mathbf{u}_p$ be the outward unit normal vector of $C \cap H_p$ in $H_p \subseteq \mathbb{E}^n$ at p and similarly define $\mathbf{u}_q$. Since C is a convex hypersurface, $C \cap H_p$ and $C \cap H_q$ have the same recession cone. By duality (Lemma 3.6.6), the closures of the outward normal cones[1] of $C \cap H_p$ and $C \cap H_q$ are equal and convex. For a Euclidean convex hypersurface, the closure of the outward normal cone

[1] By outward normal cone, we mean $-\mathcal{N}$, where $\mathcal{N}$ is the normal cone as defined in Definition 3.6.1.

intersects the recession cone trivially at $\mathbf{0}$. Thus, one of the spherical geodesics (there may only be one) starting at $\mathbf{u}_p$ and ending at $\mathbf{u}_q$ will be contained in the closure of the normal cones of $C \cap H_p$ and $C \cap H_q$ and will not intersect their recession cone. Call this curve γ.

Let H be the 2-dimensional surface in $\mathbb{E}^n$ parametrized by

$$h(t,s) = (1-t)p + tq + s\gamma(t) \text{ for } s \geq 0, t \in \mathbb{R}.$$

Notice that for any $t \in [0,1]$, because $\gamma(t)$ is not in the recession cone, the line parametrized by $\ell_t(s) = h(t,s)$ will intersect C transversally at some $s = \lambda(t) \geq 0$ and λ will smoothly vary with t.

Let $\alpha(t) = (1-t)p + tq + \lambda(t)\gamma(t)$. Note that α is a curve in C that starts at p and ends at q. Then $\alpha'(t) = \overrightarrow{pq} + (\lambda(t)\gamma(t))'$. Since γ is contained in a plane orthogonal to $\mathbf{v}$, $\langle \alpha'(t), \mathbf{v} \rangle = \langle \overrightarrow{pq}, \mathbf{v} \rangle$, which is constant in sign or vanishing if $\langle p, \mathbf{v} \rangle = \langle q, \mathbf{v} \rangle$. Thus, by Lemma 3.9.4 $L_{C,\mathbf{v}}$ is geodesically connected. $\square$

In section 3.6 we provided a topological-geometric proof of geodesic connectedness for a class of timelike convex hypersurfaces, but our proof does not work when the hypersurface is not disprisoning or is ruled by parallel null lines after splitting off a maximal non-degenerate subspace. Theorem 3.9.5 uses the results of Bartolo, Candela, and Flores to establish geodesic connectedness in the latter case. By starting with geometric hypotheses on a space-time (e.g. convexity), the types of curves in condition (ii) of Theorem 3.9.2 may arise naturally as a result of the geometry of the space-time in question.

Chapter 4

Convex functions, sectional curvature, & trapped submanifolds

4.1 Introduction

The material of this chapter is from [AK17].

In [GI01], Gibbons and Ishibashi introduce and mainly consider "space-time convex" functions on Lorentzian manifolds (M, g), or more generally, functions f satisfying

$$\nabla^2 f \geq \lambda g, \quad \lambda > 0.$$

They find examples and non-examples of such functions on regions in cosmological space-times and black-hole space-times. They show, for example, that such functions rule out closed marginally inner and outer trapped surfaces. Curvature bounds do not arise in their considerations.

The purpose of this chapter is to show that sectional curvature bounds of the form $\mathcal{R} \leq K$ are closely tied to space-time convex functions. Among the consequences:

- A natural construction of such functions.
- New domains that cannot support trapped submanifolds, namely a full neighborhood of a point q, rather than a neighborhood of q in the chronological future of q as has been considered previously, in particular by Alías, Bessa and deLira [ABL16].

In addition, we indicate connections between investigations pursued independently by various authors, including:

- Space-time convex functions [GI01].
- Comparison theorems for Lorentzian distance on domains in the chronological future of a source point or hypersurface on which the source has no Lorentzian cut points, given timelike sectional curvature controls (see for example [EGK03, AHP10, Imp12, ABL16]).
- Hessian comparisons on level hypersurfaces in exponentially embedded neighborhoods of a point or hypersurface, given a sectional curvature bound of the form $\mathcal{R} \leq K$ or $\mathcal{R} \geq K$ [AH98, AB08].

4.2 Outline of Chapter

Section 3 is an introduction to space-time convex and λ-convex functions, as defined in [GI01].

Section 4 summarizes certain theorems about Hessian and Laplacian comparisons on the Lorentzian distance function from a point or achronal spacelike hypersurface, under comparisons on timelike sectional curvature ([EGK03, AHP10, Imp12, ABL16]).

Section 5 describes results from [AH98, AB08] concerning the conditions $\mathcal{R} \geq K$ and $\mathcal{R} \leq K$ in semi-Riemannian manifolds. In particular, in [AH98] Andersson and Howard prove a comparison theorem for matrix Riccati equations which applies to the second fundamental forms of parallel families of hypersurfaces under curvature comparisons. In [AB08], this theorem is adapted to tubes around points; as an application, the geometric meaning of the bounds $\mathcal{R} \geq K$ and $\mathcal{R} \leq K$ is found by introducing signed lengths of geodesics.

In section 6, we use this framework to rule out trapped submanifolds in an exponentially embedded neighborhood of a point in a space-time satisfying $\mathcal{R} \leq K$.

4.3 Space-time convex functions

Definition 4.3.1. Given smooth functions $f : M \to \mathbb{R}$ and $\lambda : M \to \mathbb{R}$ on a semi-Riemannian manifold (M, g), f will be called *λ-convex* if the Hessian $\nabla^2 f$ satisfies

$$\nabla^2 f \geq \lambda g, \tag{4.1}$$

or equivalently,

$$(f \circ \gamma)'' \geq (\lambda \circ \gamma) g(\gamma', \gamma') \tag{4.2}$$

for every geodesic γ.

Suppose M is Lorentzian. We say f is *space-time λ-convex* if f is λ-convex for some *positive* function λ, and $\nabla^2 f$ has Lorentzian signature.

Note that this definition differs from the classical definition of convexity in that the right-hand sides of (4.1) and (4.2) need not be positive when $\lambda > 0$. Rather, controlled concavity is allowed along timelike geodesics, and is imposed in the definition of space-time convexity.

One of the simplest examples of a space-time λ-convex function is

$$f(\mathbf{x}, t) = \frac{1}{2}(\mathbf{x} \cdot \mathbf{x} - \lambda t^2), \quad (\mathbf{x}, t) \in \mathbb{E}_1^{n+1}, \tag{4.3}$$

on Minkowski space for some constant $0 < \lambda \leq 1$.

As pointed out in [GI01], the geometric meaning of space-time convexity is that at each point, the forward light cone defined by the Hessian $\nabla^2 f$ lies inside the light cone defined by the space-time metric.

Definition 4.3.1 is consistent with current Riemannian/Alexandrov usage of "λ-convex" (see [Ptr07]); and also with the definition of "space-time convex" in [GI01] except that our λ is a positive function and Gibbons and Ishibashi take λ to be a positive constant. (However, Definition 4.3.1 differs from the usage in [AB08].)

In [GI01], Gibbons and Ishibashi begin an investigation of the geometric implications of space-time convex functions. For example, they show that a space-time with a closed marginally inner and outer trapped surface cannot support a space-time convex function.

Here a *marginally inner and outer trapped surface* Σ is a spacelike submanifold of codimension 2 whose mean curvature vanishes.

Seeking examples of space-time convex functions, Gibbons and Ishibashi consider *Robertson-Walker spaces*

$$M = -I \times_f F,$$

that is, M is the product manifold carrying the warped product metric

$$-d\tau^2 + f^2 ds_F^2$$

where $I = (a, b)$, $a \in [-\infty, \infty)$, $b \in (-\infty, \infty]$, $f : I \to \mathbb{R}_+$, and F has constant sectional curvature. They ask when the function

$$-\,f^2/2 \tag{4.4}$$

is space-time convex (here we use f to denote both the warping function and its lift to M). For instance, various cosmological charts are considered on de-Sitter space dS_1^{n+1} and anti-de-Sitter space adS_1^{n+1}. One of these yields an affirmative answer: namely, the function (4.4) is space-time convex on the region

$$(0, \pi/2) \times_{\sin} \mathbb{H}^n$$

in adS_1^{n+1}.

Gibbons and Ishibashi do not consider curvature bounds when seeking examples. The perspective of space-times with curvature bounds of the form $\mathcal{R} \leq K$ suggests an alternative, namely analogs of the "square norm" ((4.3) with $\lambda = 1$). For instance, these analogs yield space-time convex functions adapted to some of the domains in de-Sitter and anti-de-Sitter space considered in [GI01].

Our theorems show that space-time convex functions arise naturally in all Lorentzian manifolds satisfying $\mathcal{R} \leq K$.

4.4 Comparisons for Lorentzian distance

Let us recall some related works concerning the Lorentzian distance functions from a point or spacelike hypersurface. All these investigations are restricted to domains containing no Lorentzian cut points of the source point or hypersurface.

1. In [EGK03], Erkekoglu, Garcia-Rio and Kupeli prove Hessian and Laplacian comparison theorems for level sets of the Lorentzian distance function from points or from achronal spacelike hypersurfaces, in two space-times M and $\widetilde{M}$. They consider corresponding timelike, distance-realizing unit geodesics in M and $\widetilde{M}$, where sectional curvatures of 2-planes tangent to the geodesics at corresponding values of the time parameter are no greater in M than in $\widetilde{M}$. Some space-time singularity theorems are given.

2. In [AHP10], Alías, Hurtado and Palmer study the restriction of Lorentzian distance from a point or spacelike hypersurface to a spacelike hypersurface satisfying the Omori-Yau maximum principle. Under constant bounds either above or below on timelike sectional (or Ricci) curvatures, they obtain sharp estimates on the mean curvature of such hypersurfaces.

3. In [Imp12], Impera studies Hessian and Laplacian comparisons for Lorentzian distance from a point, assuming timelike sectional curvatures are bounded above or below by a function of the Lorentzian distance. Estimates are obtained on the higher order mean curvatures of spacelike hypersurfaces satisfying the Omori-Yau maximum principle.

4. In [ABL16], Alías, Bessa and deLira prove non-existence results and sharp mean curvature estimates for trapped submanifolds (of arbitrary codimension), based on comparison inequalities for the Laplacian of the restriction to a spacelike submanifold of the Lorentzian distance function from a point or achronal spacelike hypersurface. They use a weak Omori-Yau maximum principle equivalent to stochastic completeness.

4.5 Curvature bounds $\mathcal{R} \leq K$, $\mathcal{R} \geq K$.

Recall that $\mathcal{R} \leq K$ means that spacelike sectional curvatures are $\leq K$ and timelike ones are $\geq K$. For $\mathcal{R} \geq K$, reverse the inequalities. (Note that $\mathcal{R} \leq K \leq K'$ does not imply $\mathcal{R} \leq K'$!)

Briefly, $\mathcal{R} \leq K$ means, as in the Riemannian case, that unit geodesics radiating from a point "repel" each other at least as much as in a space of constant curvature K, assuming the same initial conditions. However, repulsion here is meant in the *signed* sense. In particular, in the Lorentzian case, if the initial direction of variation of the geodesics is timelike, we see *negative repulsion*, that is, at least as much *attraction* as in a Lorentzian space of constant curvature K. This is explained below in Subsections 4.5.1 and 4.5.2.

4.5.1 Comparisons based at a point

Let M be a semi-Riemannian manifold, and U be the diffeomorphic image under $\exp_q$ of a star-shaped region in T_qM about O. Let $\gamma_{p,q}$ be the geodesic path in U from p to q that is distinguished by this diffeomorphism.

Define the *signed energy function* $E_q : U \to \mathbb{R}$ by

$$E_q(p) = (\operatorname{sgn}\gamma_{p,q})(\operatorname{length}\gamma_{p,q})^2, \tag{4.5}$$

where $\operatorname{sgn}\gamma$ take values $1, 0, -1$ according to whether $\gamma_{p,q}$ is spacelike, null or timelike , respectively. (Note that we use signed energy rather than signed length merely in order to preserve smoothness at q.)

Signing was shown in [AB08] to be the key to geometric understanding of the curvature bounds $\mathcal{R} \leq K$ and $\mathcal{R} \geq K$. In particular, Andersson and Howard do not consider signed distance or energy.

For a fixed choice of $K \in \mathbb{R}$ and $q \in U$, define $f_{K,q} : U \to \mathbb{R}$ by

$$f_{K,q} = \sum_{n=1}^{\infty} \frac{(-K)^{n-1}(E_q)^n}{(2n)!} = \begin{cases} E_q/2, & K = 0, \\ (1 - \cos\sqrt{KE_q})/K, & K \neq 0. \end{cases} \tag{4.6}$$

Here the argument of cos may be imaginary, yielding $\cos it = \cosh t$.

Remark 4.5.1. Note that on the lift of U to T_qM by $(\exp_q)^{-1}$, the lift of $f_{K,q}$ is the square norm if $K = 0$, and an analog if $K \neq 0$. The possible values of $(1 - Kf_{K,q})$ are 1, $\cos\sqrt{|KE_q|}$ and $\cosh\sqrt{|KE_q|}$.

Set $f = f_{K,q}$ as in (4.6), for a fixed choice of K and q. Define the *modified shape operator* $S = S_{K,q}$ to be the self-adjoint operator associated with the Hessian of f, namely,

$$Sv = \overline{\nabla}_v \overline{\nabla} f \tag{4.7}$$

where $\overline{\nabla}$ is the covariant derivative of M.

Note that the levels of f are the levels of E_q. The precise form of f was chosen for analytic convenience (following [Kar87]), so that if M has constant curvature K then S is a scalar multiple of the identity, namely

$$S = (1 - Kf)I.$$

The modified shape operator S has the following further properties: along a non-null geodesic from q, its restriction to normal vectors is a scalar multiple of the second fundamental form of the level hypersurfaces of E_q; it is smoothly defined on the regular set of E_q, hence along null geodesics from q (as the second fundamental forms are not); and finally, it satisfies a matrix Riccati equation along every geodesic from q, after reparametrization as an integral curve of $\overline{\nabla} f_{K,q}$.

The proof of the following theorem is by adapting to the set-up just described, a comparison theorem of Andersson and Howard [AH98, Theorem 3.2] that applies to exponentially embedded tubes about hypersurfaces rather than points (see Subsection 4.5.3).

We say two geodesic segments σ and $\widetilde{\sigma}$ in semi-Riemannian manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$ *correspond* if they are defined on the same affine parameter interval and satisfy $g(\sigma', \sigma') = \widetilde{g}(\widetilde{\sigma}', \widetilde{\sigma}')$. Let $R_{\sigma'}$ be the self-adjoint operator $R_{\sigma'}v = R(\sigma', v)\sigma'$, and similarly for $\widetilde{R}_{\widetilde{\sigma}'}$.

In the special case that the geodesics σ and $\widetilde{\sigma}$ are timelike, the following theorem includes comparison inequalities of Erkekoglu, Garcia-Rio and Kupeli [EGK03, Theorem 3.1] for level hypersurfaces of the Lorentzian distance from a point. However, here we are analyzing an exponentially embedded *neighborhood of a point* rather than restricting to the chronological future.

Theorem 4.5.2. *[AB08] Let M and $\widetilde{M}$ be semi-Riemannian manifolds of the same dimension and index. For $q \in M$ and $\widetilde{q} \in \widetilde{M}$, let U and $\widetilde{U}$ be diffeomorphic images under $\exp_q$ and $\exp_{\widetilde{q}}$ respectively of star-shaped regions about the origin in T_qM and $T_{\widetilde{q}}\widetilde{M}$. Let σ and $\widetilde{\sigma}$ be corresponding non-null geodesics in U and $\widetilde{U}$ respectively, radiating from q and $\widetilde{q}$.*

Identify linear operators on $T_{\sigma(t)}M$ with those on $T_{\widetilde{\sigma}(t)}\widetilde{M}$ by parallel translation to the base points, together with an isometry of T_qM and $T_{\widetilde{q}}\widetilde{M}$ that identifies $\sigma'(0)$ and $\widetilde{\sigma}'(0)$.

Suppose $R_{\sigma'} \le \widetilde{R}_{\widetilde{\sigma}'}$ at corresponding points of σ and $\widetilde{\sigma}$. Then the modified shape operators $S = S_{K,q}$ and $\widetilde{S} = \widetilde{S}_{K,q}$, as in (4.7), satisfy $S \ge \widetilde{S}$ (that is, $S - \widetilde{S}$ is positive semidefinite) at corresponding points of σ and $\widetilde{\sigma}$.

Remark 4.5.3. A more precise statement of Theorem 4.5.2 localizes at a choice of unit-speed geodesics $\sigma : [0, a] \to M$ and $\widetilde{\sigma} : [0, a] \to \widetilde{M}$, where σ and $\widetilde{\sigma}$ have no conjugate points. Specifically, we let $U \subset M$ and $\widetilde{U} \subset \widetilde{M}$ be diffeomorphic images under $\exp_q$ and $\exp_{\widetilde{q}}$ of truncated cones of the form $(0, a] \times_{\mathrm{id}} D$ and $(0, a] \times_{\mathrm{id}} \widetilde{D}$ with vertices at the origin, where D and $\widetilde{D}$ are open disks in the unit tangent spheres at q and $\widetilde{q}$ centered at $\sigma'(0)$ and $\widetilde{\sigma}'(0)$ respectively.

The following basic lemma is verified in [AB08]:

Lemma 4.5.4. *Let M be a semi-Riemannian space of constant curvature K, and U be the diffeomorphic image under $\exp_q$ of a star-shaped region in T_qM about O. Then $f_{K,q} : U \to \mathbb{R}$ satisfies*

$$\nabla^2 f_{K,q} = (1 - K f_{K,q})g.$$

Combining Theorem 4.5.2 and Lemma 4.5.4, we obtain:

Theorem 4.5.5. *[AB08] Let M be a semi-Riemannian manifold satisfying $\mathcal{R} \le K$. Let U be the diffeomorphic image under $\exp_q$ of a star-shaped region in T_qM*

about O. Assume $E_q : U \to \mathbb{R}$ satisfies $E_q < \pi^2/K$ if $K > 0$, and $E_q > \pi^2/K$ if $K < 0$. Then $f_{K,q} : U \to \mathbb{R}$ satisfies

$$\nabla^2 f_{K,q} \geq (1 - K f_{K,q}) g.$$

That is, $f_{K,q}$ is $(1 - K f_{K,q})$-convex.

4.5.2 Geometric characterization of $\mathcal{R} \leq K$, $\mathcal{R} \geq K$

The geometric characterization of Riemannian sectional curvature bounds Sec $\leq K$ or Sec $\geq K$ is given by local triangle comparisons with Riemannian space forms of constant curvature K. This is the basis of Alexandrov geometry, which extends the theory of Riemannian manifolds with sectional curvature bounds to highly singular spaces.

It turns out that this characterization by local triangle comparisons extends to semi-Riemannian manifolds *if we take lengths of geodesics to be signed.*

Recall that in a semi-Riemannian manifold, any point q has arbitrarily small normal neighborhoods U, that is, U is the diffeomorphic exponential image of a star-shaped domain in the tangent space of each of its points. There is a unique geodesic $\gamma_{p,q}$ in U between any two points $p, q \in U$.

Theorem 4.5.6 ([AB08]). *Let M be a semi-Riemannian manifold.*

1. *If M satisfies $\mathcal{R} \leq K$ ($\mathcal{R} \geq K$), and U is a normal neighborhood for K, then the signed length of the geodesic between two points on any geodesic triangle of U is at most (at least) that for the corresponding points on a* model *triangle with the same signed side-lengths in a semi-Riemannian model surface M_K with constant sectional curvature K. (For a non-degenerate triangle, M_K is uniquely determined, as is the comparison model triangle up to motion.)*

2. *Conversely, if these triangle comparisons hold in some normal neighborhood of each point of M, then $\mathcal{R} \leq K$ ($\mathcal{R} \geq K$).*

Remark 4.5.7. In [Har82] (see also [Har96]), Harris proves *global* purely timelike triangle comparisons in space-times of timelike sectional curvature bounded above. Thus, the theorem of Harris is a timelike version for Lorentzian manifolds of Toponogov's Globalization Theorem for Riemannian manifolds of sectional curvature bounded below [Top59].

4.5.3 Comparisons for parallel families of hypersurfaces

In [AH98, Theorem 3.2], Andersson and Howard prove a comparison theorem for matrix Riccati equations that applies to the second fundamental forms of parallel families of hypersurfaces of any signature in semi-Riemannian manifolds, rather than only to parallel families of spacelike hypersurfaces in Lorentzian manifolds as in Section 4.4. We give an analog in Theorem 4.5.2.

For $\mathcal{R} \leq 0$ and $\mathcal{R} \geq 0$, Andersson and Howard prove "gap" rigidity theorems of the type first proved for Riemannian manifolds with Sec ≤ 0 by Gromov [BGS85], and with Sec ≥ 0 by Greene and Wu [GW82], respectively. As applications, they obtain rigidity results for semi-Riemannian manifolds with simply connected ends of constant curvature.

We remark that while in the Riemannian case, the Riccati comparisons of [AH98] reduce to 1-dimensional equations (see [Kar87]) the semi-Riemannian case seems to require matrix-valued equations. Such increased complexity is perhaps not surprising, since semi-Riemannian curvature bounds above (say) share some behavior with Riemannian curvature bounds below as well as above.

4.6 Results

By Theorem 4.5.5 we have:

Corollary 4.6.1. *Let M be a semi-Riemannian manifold satisfying $\mathcal{R} \leq K$. Let U be the diffeomorphic image under $\exp_q$ of a star-shaped region in T_qM about O. Assume $E_q : U \to \mathbb{R}$ satisfies $E_q < \pi^2/4K$ if $K > 0$, and $E_q > \pi^2/4K$ if $K < 0$. Then $f_{K,q} : U \to \mathbb{R}$ is λ-convex with $\lambda = 1 - Kf_{K,q} > 0$ (where $f_{K,q}$ is defined in (4.5) and (4.6)) .*

Moreover, $f_{K,q}$ is space-time convex on a neighborhood of q.

Proof. By Theorem 4.5.5, $f_{K,q} : U \to \mathbb{R}$ is $(1-Kf_{K,q})$-convex. By (4.6), setting $\lambda = 1 - Kf_{K,q}$, we have

$$\lambda \;=\; \begin{cases} 1, & K = 0, \\ \cos\sqrt{KE_q}, & K \neq 0. \end{cases} \tag{4.8}$$

Suppose $K > 0$. If $E_q \leq 0$, then $\lambda = \cosh\sqrt{|KE_q|} > 0$. If $0 \leq E_q < \pi^2/4K$, then $\lambda = \cos\sqrt{|KE_q|} > 0$. Similarly for $K < 0$.

It remains to show $\nabla^2 f_{K,q}$ has Lorentzian signature in a neighborhood of q. This follows by continuity, since for a unit timelike geodesic γ satisfying $\gamma(0) = q$ we have $(f_{K,q} \circ \gamma)''(0) = -1$. ☐

In defining the second fundamental form II and mean curvature vector field H of a k-dimensional submanifold Σ of a Lorentzian manifold M, we use the convention in relativity (the opposite of that in differential geometry):

$$\overline{\nabla}_X Y = \nabla_X Y - \mathrm{II}(X, Y), \tag{4.9}$$

$$H = \frac{1}{k}\sum_i \mathrm{II}(E_i, E_i), \tag{4.10}$$

where $\overline{\nabla}$ and ∇ denote the covariant derivatives on M and Σ respectively, and $\{E_1, ..., E_k\}$ is a local orthonormal frame on Σ.

We are going to follow [ABL16] in considering submanifolds Σ satisfying the *weak maximum principle* of Pigola, Rigoli and Setti [PRS05], according to which for any smooth function u on Σ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence of points $p_n \in \Sigma$ such that

$$u(p_n) > u^* - \frac{1}{n} \quad \text{and} \quad \Delta u(p_n) < \frac{1}{n}.$$

Pigola, Rigoli and Setti proved that Σ satisfies the weak maximum principle if and only if Σ has the probabilistic property of stochastic completeness [PRS05, PRS08].

By [GI01, Proposition 8], domains carrying space-time convex functions f cannot contain closed marginally inner and outer trapped surfaces. The proof extends to the following proposition, which does not depend on the behavior of $\nabla^2 f$ on causal vectors or on the codimension, and uses the weak maximum principal to extend from closed to stochastically complete submanifolds.

Theorem 4.6.2. *Let M be a Lorentzian manifold and $f : M \to \mathbb{R}$ be λ-convex on spacelike vectors for some function $\lambda : M \to \mathbb{R}$. Then:*

(i) M contains no stochastically complete spacelike submanifold with vanishing mean curvature and on which f is bounded above and λ has positive infimum.

(ii) If $\lambda > 0$, then M contains no closed spacelike submanifold with vanishing mean curvature.

Proof. Suppose Σ is a spacelike k-dimensional submanifold with vanishing mean curvature. Let $\overline{\nabla}$ and ∇ denote the covariant derivatives on M and Σ respectively. Let II and H denote the second fundamental form and mean curvature vector field of Σ respectively. Let $u = f|_\Sigma : \Sigma \to \mathbb{R}$ denote the restriction of f to Σ.

Then for any $x \in T_p\Sigma$,

$$(\nabla^2 u)_p(x, x) = (\overline{\nabla}^2 f)_p(x, x) - g(\mathrm{II}_p(x, x), \overline{\nabla} f_p).$$

If $\{e_i\}$ is an orthonormal basis for $T_p\Sigma$, then

$$\Delta u(p) = \sum_{i=1}^{k} (\overline{\nabla}^2 f)_p(e_i, e_i) - kg(H_p, \overline{\nabla} f_p). \tag{4.11}$$

Since f is λ-convex and H vanishes, u satisfies

$$\Delta u \geq k\, \lambda|_\Sigma.$$

Thus if the Laplacian Δu is bounded below by $k \inf_\Sigma \lambda > 0$, and u is bounded above, then Σ cannot be stochastically complete. This proves (i), and (ii) follows. $\square$

Definition 4.6.3. In a causally orientable Lorentzian manifold, a spacelike submanifold M whose mean curvature vector field is causal and future-pointing is called a *weakly future-trapped submanifold.*

Remark 4.6.4. Galloway and Senovilla prove that standard singularity theorems hold in Lorentzian manifolds of arbitrary dimension with closed trapped submanifolds of arbitrary co-dimension [GS10]. They point out that such submanifolds appear to have many common properties independent of the codimension.

The significance of the following theorem lies in using sectional curvature bounds to examine geometric properties of a full neighborhood of a point q, rather than restricting to the chronological future of q.

If in the following theorem we restrict U and $\widetilde{U}$ to the chronological future of q and assume only timelike sectional curvature $\geq K$, then taking into account Remark 4.5.3, we obtain a result of Alías, Bessa and deLira ([ABL16, Corollary 4.2]).

Theorem 4.6.5. *Let M be a Lorentzian manifold satisfying $\mathcal{R} \leq K$. Let U be a domain in M that is the diffeomorphic image under $\exp_q$ of a star-shaped region in T_qM about O. Suppose that $E_q : U \to \mathbb{R}$ is bounded above and satisfies $E_q < \pi^2/4K$ if $K > 0$ and $E_q > \pi^2/4K$ if $K < 0$.*

(i) Then U contains no stochastically complete spacelike submanifolds Σ with vanishing mean curvature, and such that $\sup E_q|_\Sigma < \pi^2/4K$ if $K > 0$ and $\inf E_q|_\Sigma > \pi^2/4K$ if $K < 0$.

(ii) More generally, U contains no stochastically complete, weakly future-trapped submanifold whose mean curvature vector field H satisfies

$$HE_q \leq 0, \tag{4.12}$$

and such that $\sup E_q|_\Sigma < \pi^2/4K$ if $K > 0$ and $\inf E_q|_\Sigma > \pi^2/4K$ if $K < 0$.

(iii) Suppose

$K \neq 0$ and $U \subset \widetilde{U}$, where $\widetilde{U}$ is the diffeomorphic image under $\exp_q$ of a star-shaped region in T_qM about O, and $E_q : \widetilde{U} \to \mathbb{R}$ satisfies $E_q < \pi^2/K$ if $K > 0$ and $E_q > \pi^2/K$ if $K < 0$. Then no stochastically complete, weakly future-trapped submanifold in $\widetilde{U}$ that satisfies $HE_q \leq 0$ enters U.

Proof. By Corollary 4.6.1, the function $f_{K,q} : U \to \mathbb{R}$ as defined in (4.5) and (4.6) is λ-convex with $\lambda = 1 - Kf_{K,q} > 0$. Suppose Σ is a weakly future-trapped k-dimensional submanifold of U whose mean curvature vector field H satisfies

$HE_q \leq 0$. Let $u : \Sigma \to \mathbb{R}$ be the restriction of $f_{K,q}$ to Σ. As in equation (4.11),

$$\begin{aligned}
\Delta u(p) &= \sum_{i=1}^{k} (\overline{\nabla}^2 f_{K,q})_p(e_i, e_i) - kg(H_p, (\overline{\nabla} f_{K,q})_p) \\
&\geq k(1 - K f_{K,q}(p)) - kg(H_p, (\overline{\nabla} f_{K,q})_p).
\end{aligned}$$

Simple computation yields

$$\overline{\nabla} f_{K,q} = \begin{cases} \overline{\nabla} E_q/2, & K = 0, \\ \frac{\sin\sqrt{KE_q}}{2\sqrt{KE_q}} \overline{\nabla} E_q, & K \neq 0, \end{cases}$$

where the argument of sin can be imaginary here. The function $\sin\sqrt{KE_q}/(2\sqrt{KE_q})$ is non-negative as long as $KE_q \leq \pi^2$. Thus, $g(H_p, (\overline{\nabla} f_{K,q})_p) \leq 0$ on U since $g(H, \overline{\nabla} E_q) = HE_q \leq 0$.

Since $(1 - K f_{K,q})|_\Sigma > 0$, we conclude that u is subharmonic and satisfies the differential inequality

$$\Delta u \geq k(1 - Ku) > 0. \tag{4.13}$$

By (4.6), $u^* = \sup_\Sigma u < +\infty$. Since Σ is stochastically complete, we can apply the weak maximum principle to obtain a sequence of points $p_n \in \Sigma$ such that

$$u(p_n) > u^* - \frac{1}{n} \quad \text{and} \quad \Delta u(p_n) < \frac{1}{n}.$$

Evaluating (4.13) on p_n and taking $n \to \infty$, we obtain $1 - Ku^* = \cos\sqrt{KE^*} = 0$, where $E^* = \lim_{n\to\infty} E_q(p_n)$.

If $K = 0$, this is impossible. If $K > 0$ and $\sup_\Sigma E_q < \pi^2/4K$, then $KE^* < \pi^2/4$ and $\cos\sqrt{KE^*} > 0$, a contradiction. Similarly, if $K < 0$ and $\inf_\Sigma E_q > \pi^2/4K$, then $KE^* < \pi^2/4$ and $\cos\sqrt{KE^*} > 0$, a contradiction. Hence (ii) and (i).

Finally, suppose $K \neq 0$ and $U \subset \widetilde{U}$, where $\widetilde{U}$ is the diffeomorphic image under $\exp_q$ of a star-shaped region in T_qM about O, and $E_q : \widetilde{U} \to \mathbb{R}$ satisfies $E_q < \pi^2/K$ if $K > 0$ and $E_q > \pi^2/K$if $K < 0$.

Suppose Σ is a stochastically complete spacelike submanifold in $\widetilde{U}$. Choose a sequence $p_n \in \Sigma$ as above and let $E^* = \lim_{n\to\infty} E_q(p_n)$. By the above calculation, we know that $KE^* \geq \pi^2/4$. If $K > 0$, then $E^* \geq \pi^2/4K$ and if $K < 0$, $E^* \leq \pi^2/4K$. If $K > 0$, then $E^* = \inf_\Sigma E_q$ and if $K < 0$, then $E^* = \sup_\Sigma E_q$. Thus, in either situation Σ does not enter U. Hence (iii). $\square$

Note that for $K > 0$, the bounds on E_q in Theorem 4.6.5 affect only spacelike geodesics, and for $K < 0$, only timelike geodesics.

Remark 4.6.6. Where a weakly future-trapped submanifold Σ intersects the causal future of q, the condition (4.12), namely $HE_q \leq 0$, is immediate. Where

Σ enters the causal past of q, (4.12) implies $H = 0$. At a point p not causally related to q, (4.12) restricts H to a subcone of the cone of future directed vectors at $p:$ either $H \neq 0$ lies in a closed half-cone of the cone of future directed vectors at p, or H is null and future-pointing, or $H = 0$.

For example, in Minkowski space, consider points $\mathbf{v} \in \Sigma$ where $\mathbf{v}$ is spacelike. If $\mathbf{v}$ approaches $\mathbf{v}_0 \neq \mathbf{0}$ in the future null cone of the origin $\mathbf{0}$, these half-cones approach the causal future cone of $\mathbf{0}$; if $\mathbf{v}$ approaches $\mathbf{v}_0 \neq \mathbf{0}$ in the past null cone of 0, these half-cones approach the light ray through $\mathbf{v}_0$.

4.7 Conclusion

We have demonstrated a close connection between sectional curvature bounds of the form $\mathcal{R} \leq K$ and space-time convex and λ-convex functions ($\lambda > 0$). We have constructed new λ-convex functions. We have used these functions to find new domains that do not support trapped submanifolds.

Our goal has been to explain some viewpoints and tools, rather than to give an exhaustive treatment. We plan a more systematic treatment of results in future.

Note that the λ-convex functions considered here are based on signed energy functions. It would be interesting to identify other classes of λ-convex functions to which Theorem 4.6.2 can be applied.

Chapter 5

Future directions

5.1 Constructing convex functions on general space-times

In Chapter 3, we proved that the existence of proper strictly convex functions can be used to establish geodesic connectedness for semi-Riemannian manifolds. We used this result to prove that a timelike strictly convex hypersurface M of Minkowski space is geodesically connected by restricting distance to a tangent hyperplane in $\mathbb{E}_1^{n+1}$ to M and showing that it satisfied the conditions of Theorem 3.3.6. A few natural questions follow. In general, what are natural ways to construct convex functions on space-times? How can can the conditions $\mathcal{R} \geq K$ or $\mathcal{R} \leq K$ be used to construct convex functions?

The simplest example of a space-time and a strictly convex function that satisfies the hypotheses of Theorem 3.3.6 is Minkowski space $\mathbb{E}_1^{n+1}$ with the function $f(x^1, ..., x^{n+1}) = \frac{1}{2}((x^1)^2 + \cdots + (x^{n+1})^2)$. Notice that this function is a sum of two natural geometric functions associated to Minkowski space, $f = \frac{1}{2}(d^2 + \tau^2)$ where $d(x^1, ..., x^{n+1}) = \sqrt{(x^1)^2 + \cdots (x^n)^2}$ and $\tau(x^1, ..., x^{n+1}) = |x^{n+1}|$. The function τ is the Lorentzian distance function to $\Sigma = \{x^{n+1} = 0\}$ on $\mathbb{E}_1^{n+1}$. The function d can be viewed as distance to the origin on Σ lifted to $\mathbb{E}_1^{n+1}$ by identifying points at different moments of time by their location in the Σ.

The construction of this function could be generalized to a space-time M with a Cauchy hypersurface Σ, the Lorentzian distance τ to Σ, and some type of convex function on Σ (e.g. distance to a point) lifted to M. It would be interesting to find geometric hypotheses on M and Σ under which the function $f = \frac{1}{2}(d^2 + \tau^2)$ is proper and strictly convex on M. In addition, by restricting f to a sublevel $L_a = \{p \in M : f(p) < a\}$ and composing f with a proper convex surjective function $h : [0, a) \to [0, \infty)$, one could also show that the sublevels of f are geodesically connected.

The results of Andersson and Howard in [AH98] could be used to show that if M satisfies a curvature bound like $\mathcal{R} \leq 0$ or $\mathcal{R} \geq 0$, then the level sets of d and τ satisfy shape operator comparisons to analogous functions on Minkowski space. These comparisons could then be used to show that $f = \frac{1}{2}(d^2 + \tau^2)$ is proper and strictly convex under some additional correctly chosen hypotheses.

5.2 Constructing convex functions on convex hypersurfaces in general space-times

Another way to generalize the results in Chapter 3 is to generalize the method used to construct proper convex functions on convex hypersurfaces in Minkowski space (as in Theorem 3.6.12) to convex hypersurfaces in other space-times.

Consider a pair of smooth timelike hypersurfaces $H = \partial B$ and $H_0 = \partial B_0$ tangent at some point in M, where $B \subset B_0$ are domains in a space-time M, and where M is diffeomorphic to the normal bundle of H_0 under the exponential map. Then, one could construct a distance function $d_{H_0} : M \to \mathbb{R}$ that measures the distance away from H_0 along a spacelike geodesic emanating orthogonally from H_0. It would be interesting to find geometric hypotheses on M, H, and H_0 under which the restriction $u = d_{H_0}|_H : H \to \mathbb{R}$ of d_{H_0} to H is a proper convex function that could be used to establish geodesic connectedness for H. The Hessian of u satisfies

$$\nabla^2 u(x,x) = \overline{\nabla}^2 d_{H_0}(x,x) + \langle \mathrm{II}_H(x,x), \overline{\nabla} d_{H_0} \rangle, \tag{5.1}$$

for $x \in T_pH$. Here II_H is the second fundamental form of H, ∇ is the Levi-Civita connection on H, and $\overline{\nabla}$ is the Levi-Civita connection on M. If H_0 has positive semi-definite shape operator and M satisfies a correctly chosen curvature bound in the sense of Andersson and Howard then the first term on the right side of Equation 5.1 would be non-negative. When $M = \mathbb{E}_1^{n+1}$, H is a timelike convex hypersurface, and H_0 is a tangent hyperplane to H, this is exactly the situation in Section 3.6. In the general situation, it would be interesting to determine what kinds of geometric hypotheses might guarantee that $\langle \mathrm{II}_H(x,x), \overline{\nabla} d_{H_0} \rangle$ is positive definite, in which case H would be geodesically connected by Theorem 3.2.3.

5.3 Using distance to a timelike submanifold to rule out trapped submanifolds

In Chapter 4, we used curvature bounds to construct λ-convex functions and rule out trapped submanifolds in an exponentially embedded neighborhood of a point. In [ABL16], they use timelike sectional curvature bounds to obtain Hessian comparisons and rule out weakly future trapped submanifolds in the exponentially embedded part of the chronological future of a point. They also obtain similar results for the Lorentzian distance to an achronal spacelike submanifolds. Crucial to their approach is using future-directed timelike geodesics emanating from a point or an achronal spacelike submanifold and the sectional curvature bounds on timelike planes containing the tangent vectors of these geodesics.

Using Andersson and Howard, the conditions $\mathcal{R} \geq K$ and $\mathcal{R} \leq K$ can not only be used to obtain Hessian comparisons for the signed modified distance to a point, but also for the signed orthogonal distance to spacelike or *timelike* submanifolds. Thus, the results of [ABL16] about the Lorentzian distance to a spacelike submanifold could be generalized and applied to the positive orthogonal distance to a *timelike* submanifold. Under the correctly chosen hypotheses, such regions could rule out certain types of trapped submanifolds.

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